TO STUDY THE QUADRATURE FORMULA FOR NUMERICAL INTEGRATION

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ABSTRACT
Based on moment fitting equations, we provide effective quadrature's for the integration of polynomials over irregular convex polygons and polyhedrons. The quadrature construction strategy requires the solution of a small linear system of equations and the integration of monomial basis functions using homogeneous quadrature's with a limited number of integration points. In this study, we show that the numerical integration of polynomial functions may be performed with the same precision and efficiency as the Gauss quadrature, provided that the same points and nods are used for the determination of the polynomial's form. The locations of the chosen points are random, and the resulting deduced formulae are therefore distinct, as will be shown below and executed in order to benefit from the values of the polynomials at those sites and nods and also from their first derivatives.
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INTRODUCTION
Approximating definite integrals numerically is the goal of numerical integration. Numerical integration is useful in a wide variety of contexts. As an illustration, there are a number of well-defined functions whose anti-derivatives cannot be written in terms of fundamental functions. This operation appears in many contexts, particularly those dealing with probability and statistical testing. In addition, integro-differential equations, which are used to describe many applications in science and engineering, need a particular treatment for the integral terms (e.g. expansion, lineralization, closure ...). Therefore, not only may integrals be evaluated numerically, but special functions specified in terms of integrals can also be approximated using numerical integration.
There are two types of issues where numerical integration is necessary without sacrificing generality. The first kind involves determining the value of an integral for a known function. Because of this, the integrand may be assessed at a number of places, and numerical integration techniques can be used to determine both the best locations and the optimal number of these sites.
Secondly, differential equations, the most popular of which embody conservation principles, provide a class of problems amenable to numerical integration. Example: the well-known partial differential equation seen in process modeling and biological systems, the population balance equation, has source terms that are integrals of the solution variable (e.g. the number density function).
Quadrature is by far the most typical method for doing numerical integration. Following these three measures will result in a quadrature:
Approximate the integrand by an interpolating polynomial using a specified number of points or nodes
Substitute the interpolating polynomial into the integral
Integration

THE INTEGRAL EXPRESSIONS OF THE POLYNOMIAL FUNCTIONS
For the polynomials of first degree (Number of integral point - One):
For the polynomial functions in first degree, their integral expression is obtained as

$$I = \int_{-1}^{+1} P(\xi)d\xi = \left[\Phi_0\xi + \frac{\Phi'_0}{2}\xi^2\right]_{-1}^{+1}$$

On simplification,

$I = 2 \times \Phi_0$

In Cartesian coordinate system

$\frac{L}{2} \times 2 \times \Phi_0 = L \times \Phi_0 = (x_2 - x_1) \times \Phi\left(x = \frac{x_2+x_1}{2}\right)$

Where, Φ_0 is the value of $\Phi(x)$ at

$x = \frac{x_2+x_1}{2}$ (i.e. at $x = \frac{x_2-x_1}{2}\xi + \frac{x_2+x_1}{2}$, and $\xi = 0$)

For the polynomials of third degree (Number of integral point - Two):

On the basis of determination of a_i on the values of $\Phi(\xi)$ and $\Phi'(\xi)$ at -1.and $+1$. (i.e. at x_1 and x_2), the polynomials of third degree are defined by the expression, and their integral in the parametric formulation.

$$I = \int_{-1}^{+1} P(\xi)\, d\xi = 2a_0 + \frac{2}{3}a_2 = 2\frac{2(\Phi_1 + \Phi_{-1}) - (\Phi'_1 - \Phi'_{-1})}{4} + \frac{2}{3}\frac{(\Phi'_1 - \Phi'_{-1})}{4}$$

Thus,

$$I = 1.\times (\Phi_1 + \Phi_{-1}) - \frac{1}{3} \times (\Phi'_1 - \Phi'_{-1})$$

In Cartesian coordinate system

$$I = \int_{x_1}^{x_2} P(x)\, dx = \frac{(x_2-x_1)}{2} \times \left[(\Phi(x_2) + \Phi(x_1)) - \frac{1}{3} \times \frac{(x_2-x_1)}{2} \times (\Phi'(x_2) - \Phi'(x_1))\right]$$

Where 'Φ'' is the first derivative of the function Φ with respect to ξ and,

$$\frac{d\Phi(x)}{d\xi} = \frac{dx}{d\xi}\frac{d\Phi(x)}{dx} = \frac{L}{2}\frac{d\Phi(x)}{dx}$$

Also, the multiplication factor (x2 – x1)/ 2 appear before, represent the Jacobean of the transformation, and appears also before the derivatives with respect to ξ according to above equation.

For the polynomial functions of fifth degree (Number of integral point - Three):

In the present method, the locations of the integral points are chosen, and the derivatives are eliminated by using Gaussian fixed points. As an added bonus, if the interval's beginning and endpoints as well as its midpoint are all set to the same distance from one another, the constants a0, a2 and a4 are thus given by

$$a_0 = \Phi_0$$

$$a_2 = \frac{4(\Phi_1 + \Phi_{-1}) - 8\Phi_0 - (\Phi'_1 - \Phi'_{-1})}{4}$$

$$a_4 = \frac{-2(\Phi_1 + \Phi_{-1}) + 4\Phi_0 + (\Phi'_1 - \Phi'_{-1})}{4}$$

So, the integral of the polynomials of fifth degree (related to a0, a2 and a4) is given by,

$$I = \int_{-1}^{+1} P(\xi)\, d\xi = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4$$

On simplification:

$$I = \frac{7}{15} \times (\Phi_1 + \Phi_{-1}) + \frac{16}{15} \times \Phi_0 - \frac{1}{15} \times (\Phi'_1 - \Phi'_{-1})$$

In parametric coordinate system, it is given by,

$$I = \int_{x_1}^{x_2} P(x)\, dx = \frac{(x_2-x_1)}{2} \times \left[\frac{7}{15} \times (\Phi(x_2) + \Phi(x_1)) + \frac{16}{15} \times \frac{x_2+x_1}{2} - \frac{1}{15} \times \frac{(x_2-x_1)}{2} \times (\Phi'(x_2) - \Phi'(x_1))\right]$$

For the polynomial functions of seventh degree (Number of integral point - Four):

If P(ξ) is a polynomial function of seventh degree, then its exact integral in the parametric coordinates (related to a0, a2, a4 and a6) is given by,

$$I = \int_{-1}^{+1} P(\xi)\, d\xi = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \frac{2}{7}a_6$$

Otherwise, can be shown as,

$$I = W_1 \times (\Phi_1 + \Phi_{-1}) + W_2 \times (\Phi_{1/6} + \Phi_{-1/6}) + \overline{W_1} \times (\Phi'_1 - \Phi'_{-1}) + \overline{W_2} \times (\Phi'_{1/6} - \Phi'_{-1/6})$$

On replacement of Φ_i and Φ'_i , we get

$$I = W_1(2a_0 + 2a_2 + 2a_4 + 2a_6) + W_2\left(2a_0 + \frac{2}{6^2}a_2 + \frac{2}{6^4}a_4 + \frac{2}{6^6}a_6\right) + \overline{W_1}(4a_2 + 28 + 16a_6) + \overline{W_2}\left(\frac{14}{6^1}a_2 + \frac{2}{6^3}a_4 + \frac{2}{6^5}a_6\right)$$

Here, above two representations will be same. The following system of equations becomes

$$\begin{cases} W_1 + W_2 = 1 \\ W_1 + \frac{1}{6^2}W_2 + 2\overline{W_1} + \frac{2}{6}\overline{W_2} = \frac{1}{3} \\ W_1 + \frac{1}{6^4}W_2 + 4\overline{W_1} + \frac{4}{6^3}\overline{W_2} = \frac{1}{5} \\ W_1 + \frac{1}{6^6}W_2 + 6\overline{W_1} + \frac{6}{6^5}\overline{W_2} = \frac{1}{7} \end{cases}$$

For the polynomial functions of ninth degree (Number of integral point - Five):

Using the above similar process, we obtain the polynomial functions of ninth degree (five integral points).

$$\begin{cases} W_1 + 2W_2 + 2W_3 = 2 \\ W_2 + \frac{1}{4}W_3 + 2\overline{W_1} + \overline{W_2} = \frac{1}{3} \\ W_2 + \frac{1}{4^2}W_3 + 4\overline{W_1} + \frac{1}{2}\overline{W_2} = \frac{1}{5} \\ W_2 + \frac{1}{4^3}W_3 + 6\overline{W_1} + \frac{3}{16}\overline{W_2} = \frac{1}{7} \\ W_2 + \frac{1}{4^4}W_3 + 8\overline{W_1} + \frac{3}{16}\overline{W_2} = \frac{1}{9} \end{cases}$$

With the use of above one to five integral points, we can determine the weights Wi and W¯_i of functions Φ(ξ_i) and Φ'(ξ_i) for various points and their selected positions ξ_i.

Similarly for two dimensions, the quadrature formulas for P=P(ξ,η) can be obtained (from integrating with respect to ξ and hence with respect to η). The limit integral of the polynomial function in the parametric formulation (square area), can be formulated by

$$I = \int_{-1}^{+1}\int_{-1}^{+1} P(\xi,\eta)\, d\xi d\eta = \int_{-1}^{+1}\left[\sum_i W_i P(\xi_i,\eta) + \overline{W_i} P'_{,\xi}(\xi_i,\eta)\right] d\eta$$

$$= \sum_j \left[W_j \left[\sum_i W_i P(\xi_i,\eta_j) + \overline{W_i} P'(\xi_i,\eta_j)\right] + \overline{W_j}\left[\sum_i W_i P'_{,\eta}(\xi_i,\eta_j) + \overline{W_i} P'_{,\xi\eta}(\xi_i,\eta_j)\right]\right]$$

Or can be written as

$$I = \sum_i \sum_j \left[W_i W_j P_{ij} + \overline{W_i} W_j P'_{ij,\xi} + W_i \overline{W_j} P'_{ij,\eta} + \overline{W_i}\overline{W_j} P'_{ij,\xi\eta}\right]$$

Furthermore, in three dimensions

$$I = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} P(\xi,\eta,\zeta)\, d\xi d\eta d\zeta$$

can be given by

$$I = \sum_i \sum_j \sum_k \left[W_i W_j W_k P_{ijk} + \overline{W_i} W_j W_k P'_{ijk,\xi} + W_i \overline{W_j} W_k P'_{ijk,\eta} + W_i W_j \overline{W_k} P'_{ijk,\zeta} + \overline{W_i}\overline{W_j} W_k P'_{ij,\xi\eta} + W_i \overline{W_j}\overline{W_k} P'_{ijk,\eta\zeta} + \overline{W_i} W_j \overline{W_k} P'_{ijk,\xi\zeta} + \overline{W_i}\overline{W_j}\overline{W_k} P'_{ij,\xi\eta\zeta}\right]$$

where, Φ_{ij} and $\Phi'_{ij,\xi}$ are respectively shows the ordinates of functionΦat the point(ξ_i, η_j) and the first derivative with respect to variable ξ, and similarly for the second derivatives of the remaining ordinates.

RESULTS AND DISCUSSION

The deformation-displacement vector $\lfloor B \rfloor$ (For the element beam, with the inertia moment I, length L and elasticity modulus E) is defined as

$$\lfloor B \rfloor = \left\lfloor -\frac{6}{L^2} + \frac{12}{L^3}x \quad -\frac{4}{L} + \frac{6}{L^2}x \quad \frac{6}{L^2} - \frac{12}{L^3}x \quad -\frac{2}{L} + \frac{6}{L^2}x \right\rfloor$$

while the value of stiffness matrix [k] can be determined as

$$[k] = \int_0^L \lfloor B \rfloor^T EI \lfloor B \rfloor dx = EI \int_0^L \lfloor B \rfloor^T \lfloor B \rfloor dx$$

Now, we can calculate the stiffness element k11 by the use of two integral points, we obtain

$$k_{11} = EI\frac{L}{2}\left[1.\times(\Phi_1 + \Phi_{-1}) - \frac{1}{3}\times\frac{L}{2}(\Phi'_1 - \Phi'_{-1})\right]$$

The stiffness element k11, using the following expressions,

$$\Phi(x) = \left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)^2, \text{and } \Phi'(x) = \frac{24}{L^3}\left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)$$

which give

$$\Phi_1 = \frac{36}{L^4}, \Phi_{-1} = \frac{36}{L^4}, \Phi'_1 = \frac{144}{L^5} \text{ and } \varnothing'_{-1} = \frac{-144}{L^5}$$

reduces to

$$k_{11} = 12\frac{EI}{L^3} = exact\ solution$$

In the similar way, the rest of the matrix's components may be verified with little effort and conclude that all that's left are the constitution elements of the stiffness matrix for the two-nod beam element, can be derived from integration.

$$[k] = \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ Sym & & & 4L^2 \end{bmatrix}$$

Exact results would be obtained for all the stiffnesselements, and using also the Gauss quadrature, putting$x = \frac{L}{2}(\xi + 1)$and $\xi = \pm\sqrt{3}$, and the between difference that the vector$\lfloor B \rfloor$is actually computed at$\xi = \pm 1$instead of$\xi = \pm\sqrt{3}$, the fact which conducting to compute the stresses and strains at the same points which represents an inconvenience with Gauss quadrature and in this contributed quadrature is actually modified.

For the polynomial function (in cubic form)

$$\Phi(x) = A_0 + A_1x + A_2x^2 + A_3x^3$$

and the direct integral from 0 to L is given by

$$\int_0^L \Phi(x)\,dx = A_0L + \frac{1}{2}A_1L^2 + \frac{1}{3}A_2L^3 + \frac{1}{4}A_3L^4$$

Therefore, we get (By the above numerical formula)

$$I = \frac{L}{2}\left[\left(\Phi(L) + \Phi(0)\right) - \frac{1}{3}\frac{L}{2}\left(\Phi'(L) - \Phi'(0)\right)\right]$$

Thus, we have

$$\Phi(L) + \Phi(0) = 2A_0 + A_1L + A_2L^2 + A_3L^3$$

$$\Phi'^{(L)} - \Phi'(0) = 2A_2L + 3A_3L^2$$

On replacement this in the formula above, we obtain

$$I = A_0L + \frac{1}{2}A_1L^2 + \frac{1}{3}A_2L^3 + \frac{1}{4}A_3L^4$$

It should be noted that the idea of quadrature would also be applicable to this proposition with respect to the ordinates of the polynomial functions themselves, in which the derivatives terms, only represent contributions to meet the integration exact results. With this method, the exact results would be obtained thus for polynomial functions with degrees less than the odd degree of and approximately for every other mathematical function.

CONCLUSION

For the stresses and strains, as well as the components of the deformation-displacement matrices in solid mechanics, the analysts must generally actually obtain the results at the nodal levels of the elements because extrapolation functions, particularly at the edge nods, make it impossible to provide accurate results. Additionally, the suggested integration formulae for polynomials, which were intended specifically for the developers of finite elements, might be more advantageous for analysts who work with finite elements as well as for the integration of polynomials in general. The intervention of the first derivatives ordinates here simply means that the complete freedom of the choosing points positions, and their contributions in this effect to get exact numerical integration of polynomial functions and avoid actually the required points' positions in the class, are implied by the same numerical results obtained using the developed formulas and would otherwise be obtained using Gauss quadrature and exact direct integration ones.

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