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A hand book on Dedekind's Cut

JV'n Prof. (Dr.) Shobha Lal

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“As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the ideas of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continuously but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. ... This feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep mediating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis”.

- Dedekind

" A time will come when all living bodies including Monkeys and Cows, will start to speak numeric language and the language would be recognized through Real Analysis concept - mathematical language, and frankly telling, will start interaction in numeric language for their food and survival, then the importance of all non terminating digits and fractions would be more and a specific like cut rather than what told by Dedekind, will come to be invented in the world of Mathematics and Computing, as a Professor of Mathematics and Computing at Jayoti Vidyapeeth Women's University,Jaipur, Rajasthan, India, I feel so."

-Shobha Lal

26 November,2020

Author Speaks

In the book of Mathematics or Computing it is difficult to claim the originality of theorems, corollary and results, as different scholars and learned Authors have written a lot. Almost present form of book is the hand book as ready reckoner for the students of UG/PG/Research for the students of all Indian Universities. Since long time the attention of Mathematical Scientists have been directed towards the understanding of the real numbers systems processes occurring in the evolution any certain decision in Daily life . Only recently it has become apparent that the late stages of evolution of a digits is uniquely characterized from the energetic point of view by gravitational and rotational energy too including other aspect of Research, and that the most violent and energetically relevant moments in the life of a star indeed take place after the exhaustion of the nuclear sources of energy has occurred.cts ensuing from gravitational collapse, appear to be more. People talk about Newton, Einstein but less about a heroic scientist who invented zero and in the fifth century established the concept of gravitational pulling by the earth though existence of Newton is supposed in fourteen century.

Then the importance of Real number systems come as - Gravitational interaction either slowly generating the luminosity of white dwarfs through their continuous contraction, or producing in the accretion process in X-ray source or, again completely determining the physical process during gravitational collapsed object ensuing from gravitational collapse, appear to be more and more the back- bone, the only fundamental field theory, of this drastically new domain of physics. In this sense the in-depth analysis of a relativistic theory of gravity has become in the recent years not only much more easy due to the existence in nature of collapse standing of very large number of objects, but it was impossible to understand with specific nomenclature of number system to be counted in real number systems only, but also much more relevant to our understanding of very large number of physical processes, inherent with the numbers and number lines . If we turn to the long range program me of research we have seen in the recent years the preparation of an entire new chapter of even, Astrophysics , algebra and others what we could call". Burst Astronomy"- it would had not seen days of the light without fractional understanding. In another aspect one can see- gravitational of neutrino bursts, of possible associated electromagnetic radiation are all rapidly improving in sensitivity and sophistication while in the theoretical field basic advances are made in the analysis of fully relativistic, to be expressed in the real number systems only, short time phenomena. In the next few years we should be able to reach in all these different experimental fields the limits of detachability theoretically predicted, what Dedekind has forecasted. The direct observation through different techniques of the moment of gravitational collapse appear to be of the great interest. More we learn of the physics characterizing the configurations of equilibrium of cold catalyzed matter, the more we see this need of processes of gravitational collapse to occur under a variety of regions. It is also clear that it is unavoidable that all collapsed objects we are today observing either in pulsars or in binary X-ray source s had to be formed through this processes.

The observation of the moment of gravitational collapse will disclose, among others, one of the most fundamental predictions of general relativity namely that gravity, as any of her long range interaction has to propagate with a finite velocity equal to the speed of light and that gravitational energy can be carried by waves.

The direct technological advance brought by these observations will most likely be very limited the influence however for our understanding of nature will certainly be enormous. This brings us to another domain of physics also dominated by a fully relativistic theory of gravity: Cosmology. The greatest sources of our research on gravitational collapse will be reached if not only we will be able to describe using different technique and detailed theoretical work the physical process occurring in this very short phenomena but if we will be able to apply to cosmology the enormous amount of knowledge we are acquiring in this research. Recent development s in Astronomy and Astrophysics , that is, the discovery and study of quasi-stellar sources, of explosions in galactic nuclei, of strong extra solar X-ray sources , and of pulsating radio source- have lead to a reawakening of interest in the possible roles in nature of relativistic systems with strong gravitational fields. Thus far, this interest lies concentrated largely on roles which relativistic stars might possible play in various astronomical situations. As a result, such effort has been expanded on studies of structures and stabilities of the relativistic stars. The modern form of the problems of gravitational collapse dates back to the work of Chandershekhar and Landau(in 1930's) who firs showed that the normally accepted physical laws do not permit the existence of any cold static equilibrium states(whether in planetary, white dwarf or even neutron star form) for bodies more than one or two times as massive as the sun, since the formation of stars up to several tens times more massive than this occurs frequently within our galaxy, and since such massive stars burns up their nuclear energy reverses extremely rapidly by comparison with cosmological times scale, it is hard to avoid the conclusion that many stars within our own immediate neighborhood must already have reached the stage of being faced with runaway gravitational collapse. This poses on our hand the theoretical problems of understanding what goes on in such a collapse and on other hand the observational problem of de recognizing and detecting the collapsed objects which presumably exist around about us.

These questions have given rise to such wide spread and intense interest and activity in the last few years that it is hard to understand why, apart fro a very small number of individuals (including most notably J.R.Oppenheimmer and J.A Wheeler) . Very few physicists gave any attention at all such phenomena prior to 1960, and why they were neglected even longer by observational Astronomers who tended to brush aside collapsed objects as figments of the theoretician's imagination. Today, however, the situation has been revolutionized. On the one hand every substantial process on the basis of Einstein's General Theory of Relativity- has been made on the theoretical collapse problem during the last decade, and on the other hand Astronomers have tended to take theoretical predictions much more seriously since the 1968 discovery of Pulsars (which lead in particular to the configuration of the standing prediction of existence of a neutron star at heart of the "Crab Nebula")in the form of Real number system too

The other massive line of investigation deriving from Penrose 's paper has been the study of what are today known as "BLACK HOLE", that is to say extended regions of space-time (containing the singularities in their interiors) where the gravitational field, although finite, is sufficiently strong to prevent the escape even of particles moving with the velocity of light . In particular case of exactly spherical collapse, the formation of Black hole bounded by a horizon at the Schwarzschild radius was already well known . \

It evident fact that gravitational collapse and its evidence in the term of White dwarf, Neutron Stars and Black holes(Rotating and Non rotating) in our galaxy exists.

Adventure of Real Numbers can be seen in a detailed study of the properties of time –like and Null geodesics, specially of stable circular orbits, in the charge free Kerr metric have been presented by Bardeen, has been made .In the case of Bardeen, the three direct orbits radius of marginally stable circular, radius of marginally bound orbit and radius for circular photon orbit, all seems to coincide with the event horizon in extreme Kerr black hole . He has also obtained the three orbits rms= 6m, rmb=4m and rph=3m for schwarzs child metric. But in his case there are no circular orbits corresponding to rms=6m rmb=4m, and rph=3m, conciding with event horizon. To have to put the fact that in addition to the stable circular orbits obtained by Bardeen, there exists a circular orbit out side the ergo sphere of the Kerr black hole where all these direct orbits(marginally stable, the marginally bound and the circular photon orbits) coincide. I have tried my best to show first of in the world the storage of energy due to celestial objects due to gravitational collapse is possible in the Skylab .Interestingly, convergence of null rays, orbit of the emitter and photon orbit s, frequency shift and fluctuation periods have been discussed in details besides the following conclusions real asymmetric zones are required

"Lastly, it appears that the circular orbits near the event horizon of the perturb bed metric can be able to explain the formation and nature of rings round the Saturn and other planet s in our galaxy". A base line concept may be seen as

" In about five billion years the sun will have consumed so much of its Hydrogen in thermonuclear reactions that it will evolve to a star of this type known as" Red giant". Standard theory predicts that the sun will grow to some 250 times of its present diameter of 8, 50,000 miles, devouring Mercury, Venus and Probably the earth in the process " and my opinion is that

"Black hole is useful for all existing creatures in the Universe, as it maintains force of equilibrium amongst different celestial bodies of our Galaxy, don't fear but must love Black hole, Miracly, in absence of sun ,in the Universe, Black hole would be saving you and your lives , in the absence of sun life is possible since master energy saver Black hole would continue even there is no halting probability of surviving the sun after five billions years, we can control process of nuclear fission and fusion inside the sun 24 by7..".In my opinion whatever be the achievement of Astrophysics and Astronomy equal credited should be given to this Bihar(Patna) based Indian Scientist in which there was huge role of Number system and Real Analysis.

Lastly,I shall consider my humble effort successful if the hand book becomes useful to the global readers and the teachers and if it is well received by them. Any suggestion or improvement will be gratefully appreciated.

I am indebted to many persons and several organizations for helping in preparing the manuscript. Firstly,, I express my sincere thanks to my teacher and guide Dr. S.S.Prasad, former professor of mathematics and principal, T.P.Verma College Narkatia ganj, Bihar, India. My thanks are also due to Inter University Centre for Astronomy and Astrophysics (IUCAA) at pune where my freelance visit helped me to see some practical aspect of subject matter and utilization of numerical aspects . In this connection, I must register my sincere thanks to the scientists of Indian Space Research Organization (ISRO)

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Dr. S.Lal.

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Indexing

DEDEKIND'S THEORY

1.1. Overview:

In this Hand book we shall learn about the introduction of the real number system which occupies the fundamental position in any discussion of mathematical discipline, much less the study of mathematical analysis; nay, In Modern Era the study of mathematics starts with the system of real numbers. For the construction of real numbers system we shall assume familiarity with rational number and their basic properties. But for the sake of continuity we would like to present in brief the extension from the set of positive integers to rational number system. This extension has been found necessary to obviate difficulties encountered in the solution of algebraic equations. For example, consider the solution of the equation $a + y = b$ where a and b are positive integers and $a > b$. We know that so long as we restrict ourselves to the set of positive integers, this equation has no solution. We are thus led to the introduction of negative integers and say that the equation $a + y = b$ has a solution given by $y = b - a$ even when $a > b$. Again, consider the solution of the equation $ay = b$ where a and b are integers and b is not a multiple of a. As before, if we restrict ourselves to the set of integers, this equation has no solution. But in order to have a solution of this equation we introduce rational numbers. Thus we say that the equation $ay = b$ has a solution given by $y = \frac{b}{a}$ even when b is not a multiple of a. In this way we build up a bigger class, beginning with the set of positive integers to the set of integers which include the former and thence the set of rational numbers that include the set of integers both as positive and negative as a special case.

Here, we want to assert that a bigger class still exists. So far we have presented the extension from the set of positive integers to the set of rational numbers through the solution of algebraic equations. But there are equations whose solutions do not lie in the set of rational numbers. For example, consider the solution of the equation$y^2 = 2$. From this we get$y =$

$\sqrt{2}$. It will be shown that $\sqrt{2}$ is not a rational number. Thus we come across certain numbers which are not rational numbers. We call them irrational numbers.

The aim of this Exercise is to characterize these numbers analytically. We shall study in this here how Dedekind, a veteran scientist and German Mathematician (1831-1916) characterised these numbers by means of sections or cuts of rational numbers. This Scientist took the notion of cut as the basic notion in his theory of real numbers in Realm Analysis. The concept 'real number' synonymous with 'cut'; a cut produced by a rational number was called a real rational number and one not so produced was called a real rational number and one not so produced was called a real irrational number.

Readers are supposed to go through the other concept related to this also .Note please that there are two more important theories regarding the introduction of real numbers: This first one is that of Cantor (1845-1918, German) and the 2^{nd} other is that of Weierstrass (1815-1897, German). Notable fact is that Cantor took the notion of Cauchy sequence fundamental in his theory of real numbers by postulating that a real number was nothing but a class of equivalent Cauchy sequences of rational.

Fact is that a Cauchy sequence of rationals convergent to a rational number was called a real rational number and a Cauchy sequence of rationals which did not converge to a rational number was called an irrational number earlier.

A noted theory as –Weierstrass took the notion of nested intervals and propounded his theory of real numbers by prescribing that a real number was nothing but a class of nested intervals of rationals in Real number system. Basically, a class of nested intervals closing upon a rational was called a real rational number and one not so closing upon a rational number was called a called a real irrational number whereas there is intention to lead towards other concept also .

Cantor's and Weierstrass method of introducing real numbers, though they will be referred to in their proper places but simply to discuss how Dedekind's introduced the theory of real numbers through the sections of rational numbers-as a whole Real number system.

Readers should note that the system of real numbers differs from the system of rational numbers in one very important respect namely that the system of real numbers is complete whereas the system of rational numbers is not complete- a brain storming fact is for discussion here.

It means that if we sectionise the collection of rational numbers is the sense of Dedekind, then we shall get a new number different from rational numbers and that is our real number. But if we setionise the collection of real numbers, then we shall not get a new number other than a real number. That is, Dedekind's cut of the system of real numbers is always a real number.

Similarly in the sense of Cantor there are convergent sequences of rational numbers such that their limit is not always a rational number.

But the limit of a convergent sequence of real numbers is always a real number. Thus judging from either of the two methods we find that there are gaps in the system of rational numbers but there are no gaps in the system of real numbers. We express this fact by saying that the system of real numbers is complete but the system of rational numbers is not complete. Thus we find that the system or the set of real numbers not only includes the set of rational numbers as a subset but also retains all those properties which are possesses by rational numbers and moreover it possessesone more additional property namely that the set of real numbers is complete whereas the set of rational numbers is not. As a matter of fact, between any two rational numbers there exists an infinite number of rational numbers. Thus the system of rational numbers is dense and it appears that it is a complete system. But it is not so. In other words, the set of real numbers is a complete ordered field, but the completeness property is absent form the set of rational numbers; it is simply an ordered field. In this sense we can say that the set of real numbers is richer than the set of rational numbers. Consequently the set of real numbers comes to play a dominant role in mathematics and in particular in analysis. In fact many concepts in topology are abstractions of properties of the set of real numbers. In the next chapter it is our attempt to characterise the properties of the set of real numbers and to place them in a general setting so that the theorems proved for one-dimensional point-set of real numbers may admit of immediate generalisations.

But before doing so, we would like to recall inherent properties and deficiencies of the rational number system.

1.2. DEFINITION

Rational Number – A fraction of the from $\frac{p}{q}$ where p and q are integers in their lowest terms and $q \neq 0$ is called a rational number.

Froexample $\frac{1}{2}, \frac{4}{9}, \frac{-2}{3}$, …. are rational numbers.

We shall next show that the set of rational numbers is a field w.r.t. usual addition and multiplication, and also it is linearly ordered. For this, we shall recall the following field axioms and order axioms.

1.3. THE FIELD AXIOMS

A non-empty set $(F, +, \cdot)$ together with the two operations + and respectively called addition and multiplication, is called a field if the following conditions are satisfied:

Let $a, b, c \in F$ be arbitrary.

I. Laws of addition :

(ii)$a + b \Rightarrow F$(Closure law)

(ii)$a + (b + c) = (a + b) + c$ (associative law)

(iii)There exists an identity element denoted by 0 such that for each $a \in F$, $a + 0 = 0 + a = a$ (Existence of identity element)

(iv)For each $a \in F$, there exists an element a', called the inverse of a such that $a + a' = a' + a = 0$ (Existence of inverse)

The additive inverse of a is generally denoted ny$(-a)$.

(v)$a + b = b + a$ (Commutative law)

II. Laws of multiplication :

(vi)$a, b \in F$(Closure law)

(vii)$a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associative law)

(viii)For each $a \in F$, there exists an unity element denote by '1' such that $a \cdot 1 = 1 \cdot a = a$ (Existence of unity)

(ix)For each non-zero $a \in F$, there exists an elementa^{-1}, called the multiplicative inverse of a such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. (Existence of inverse)

(x)$a \cdot b = b \cdot a$ (Commutative law)

III. Distributive laws

(xi) $a \cdot (b + c) = a \cdot b + a \cdot c$

(xii) $(b+c)\cdot a = b\cdot a + c\cdot a$

We shall not dwell on the details of deducing from the definition of a field all of the familiar algebraic facts and the rules governing manipulations. The facts and rules are part of the study of moderns abstract Algebra.

1.4. THE FIELD AXIOMS

A field is called linearly ordered if it has additional structure namely a relation "<" which satisfies the properties of 'less than' as used in the real number system. Thus a field F is called a linearly ordered field or simply an ordered field if there is a relation '<,' which establishes an ordering among the members of F and which satisfies the following axioms :

(a) Exactly one of the relations $x = y, x < y, x > y$ holds.

It has to be noted that $x > y$ means the same as $y < x$.

(b) $If\ x < y$, then for every z in F, we have $x + z < y + z$.

(c) $If\ x, y, z \in F$ and $x > y, z > 0$, then $xz > yz$.

(d) $If\ x, y, z \in F$ and $x > y, y > z$, then $x > z$.

1.5. THE SET OF RATIONAL NUMBERS IS A FIELD

Let Q be the set of rational numbers, that is, numbers of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$, together with the two operations of addition and multiplication.

We are going to show that the set of rational numbers Q is a field w.r.t. addition and multiplication. For this, we observe that the following laws of addition, multiplication and distributive laws are obeyed.

I. Laws of addition

(i) If $\frac{a}{b}$ and $\frac{c}{d} \in Q$, then $\frac{a}{b} + \frac{c}{d}$ which is = $\frac{ad+bc}{bd}$

(a rational number also $\in Q$.

Therefore the set of rational numbers is closed w.r.t. addition.

(ii) If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Q$, then

$$\left\{\frac{a}{b} + \frac{c}{d}\right\} + \frac{e}{f} = \frac{ad+bc}{bd} + \frac{e}{f} = \frac{adf + bcf + bde}{bdf}$$

$$\frac{a}{b} + \left\{\frac{c}{d} + \frac{e}{f}\right\} = \frac{a}{b} + \frac{cf+de}{df} + \frac{e}{f} = \frac{adf+bcf+bde}{bdf}$$

$$\therefore\ \left\{\frac{a}{b} + \frac{c}{d}\right\} + \frac{e}{f} = \frac{a}{b} + \left\{\frac{c}{d} + \frac{e}{f}\right\}.$$

Thus the associative law is satisfied.

(iii) The identity is zero, for $\frac{a}{b} + 0 = \frac{a}{b}\cdot$

(iv) The inverse of $\frac{a}{b}$ is $\left[-\frac{a}{b}\right]$, for $\frac{a}{b}+\left\{-\frac{a}{b}\right\}=0$

(v) Also, $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}\cdot$

That is, the commutative law is satisfied.

Thus we find that the set of rational numbers is an abelian group under addition.

II. Laws of multiplication

Let Q^* be the set of non-zero rational numbers.; that is, Q^* is the set Q with zero excluded. It can be shown as in the previous section (I) that

(vi) The product of two rational numbers is a rational numbers; Hence if $,\beta \in Q^*$, then $\alpha \cdot \beta \in Q^*$.

(vii) The multiplication of rational numbers is associative. Hence if $\alpha, \beta, \gamma \in Q^*$, then $\alpha \cdot (\beta \cdot \gamma)=(\alpha \cdot \beta) \cdot \gamma$

(viii) The identity of Q* is $1 \in Q$, for $a \cdot 1=1 \cdot a=a$, for $a \in Q^*$.

(ix)The inverse of $a \in Q^*; a \neq 0$ is $\frac{1}{a}(or\ a^{-1}) \in Q^*$, for

$$\alpha \cdot \left[\frac{1}{a}\right]=\left\{\frac{1}{a}\right\} \cdot a=1$$

(x) The multiplication in Q^* is commutative; that is, if $\alpha, \beta \in Q^*$, then $\alpha \cdot \beta=\beta \cdot \alpha$

Thus we find that the set of non-zero rational numbers forms an abelian group under multiplication.

III. Distributive laws :

If $\alpha, \beta, \gamma \in Q$, then obviously

$\alpha \cdot (\beta+\gamma)=\alpha \cdot \beta+\alpha \cdot \gamma$

$(\beta+\gamma) \cdot \alpha=\beta \cdot \alpha+\gamma \cdot \alpha$

Thus the distributive laws are satisfied.

Hence taking I, II,and III together we conclude that the set of rational numbers Q is a field w.r.t. addition and multiplication.

1.6. PROPERTIES OR CHARACTERISTICS OF RATIONAL NUMBER SYSTEM

Let Q be the set of rational numbers.

The set Q possesses the two fundamental laws corresponding to (i) Laws of order and (ii) Algebraic structure.

In the first case, we shall deal with the inequalities between two rational numbers and in the second case, we shall show that the set of rational numbers is a field and this we have shown in the preceding article.

Consequently we discuss here the order structure of the set Q. In this connection we shall use the usual inequality symbol '$>$' to denote the relation 'greater than'. Thus $a > b$ shall mean that a is greater than b.

We enumerate the following laws of order subsisting in the set of rational numbers Q.

01. Law of trichotomy : Given any two rational numbers a, b one and only one of the following holds :

$a > b, a =, b > a$

02. Transitivity :if a, b, c be three rational numbers such that a>b, then $a > b$ and $b > c$, then $a > c$.

This property is expressed by saying that the order relation is transitive.

03. Order addition : If a, b, c be three rational numbers such that $a > b$, then $a + c > b + x$.

04. Order multiplication : If a, b, c be three rational numbers such that $a > b$, then $c > 0$, then $ac > bc$.

It is because of the properties $01 - 04$ that we say that the field of rational numbers is ordered.

1.7.

1.8. THEOREM

There are infinity of rational numbers between any two different rational numbers. [P.U. 57H,61H, 66H; Bhag. 92H]

Proof : Let a and b be any two different rational numbers and let

$$a < b. \quad \text{...(1)}$$

Suppose that k is any positive rational number. Then since $k > 0$, we have,

$$ak < bk \quad \text{...(2)}$$

$\Rightarrow$ $a + ak < a + b$k; adding a to both sides

$\Rightarrow$ $a(1 + k) < a + b$

$$\therefore \quad a < \frac{a+bk}{1+k} \quad \text{...(3)}$$

Again adding bk a to both sides of (1), we get

$$a + bk < b + bk \Rightarrow a + bk < b(1 + k)$$

$$\therefore \quad \frac{a+bk}{1+k} < b. \quad \text{...(4)}$$

Hence from, (3) and (4), we get

$$a < \frac{a+bk}{1+k} < b. \quad \text{...(5)}$$

Now $\frac{a+bk}{1+k}$ is a rational number which lies between a and b and this is true for every positive rational number k. But k is at our choice and we can assign infinity of values to k. Hence corresponding to infinite values of k we find that can pick up infinity of values. Each value is a rational number which lies between a and b.

Hence we prove the theorem.

1.9. DENSITY THEOREM

Theorem : The system of rational numbers is dense everywhere r, there are infinity of rational numbers in the neighborhood of r.

We proceed as follows:

Let r be any rational number and let h be any small positive rational number, Then r-h and r+h both are rational numbers and the set of all rational points between (r-h,r+h) defines the neighborhood of r.

We know that there are infinity of rational numbers between any two different rational numbers. Hence there are infinity of rational numbers between two rational numbers r-h and r+h. That means that there are infinity of rational numbers in the neighbourhood of the rational number. But r is any rational number. Thus the system of rational numbers is dense everywhere.

1.10. Based ARCHIMEDEAN PROPERTY OF RATIONAL NUMBERS

Theorem : if a and b be two positive rational numbers such that $a < b, a \neq 0$,then there exists a positive integer n such that nb>a.

The effect of the theorem is to show that however small b may be, we can find a positive integer n such that nb>a.

Proof : We have to prove here that there is some $n \in Z$ such that $\mathrm{nb} > a$. Here we take Z to be the set of positive integers.

We suppose on the contrary. This means that we have to prove now that $\mathrm{nb} < a$ for all $n \in Z$.

Therefore we suppose that $\mathrm{nb} < a$ for all $n \in Z$.

Then $$\mathrm{nb} < a \Rightarrow n\frac{\mathrm{b}}{a} < 1$$

$$\Rightarrow \frac{\mathrm{nb}}{\mathrm{a}} < m \ for\ all\ m \in Z$$

$$\Rightarrow \frac{\mathrm{nb}}{\mathrm{m}} < a$$

$$\Rightarrow \frac{\mathrm{n}}{\mathrm{m}} < \frac{\mathrm{a}}{b} \qquad ...(1)$$

But $\frac{\mathrm{n}}{\mathrm{m}}$ is positive rational number say x. Hence

(1) $\Rightarrow \mathrm{x} < \frac{\mathrm{a}}{b}$ for all $x \in Q^+$.

Where Q^+ denotes the set of positive rational numbers and this is false. Thus on the supposition that $\mathrm{nb} < a$for all $n \in Z$ we arrive at a contradiction.

Thus we prove the theorem.

1.11. EXISTENCE OF NON-RATIONAL NUMBERS.

We shall take two examples to show that there are certain numbers which are not rational numbers.

EX.1. Prove that $\sqrt{2}$ is not a rational number.

[P.U. 51H;VBU.55H, 63H, 65H; Bhag. 65H;R.U. 65H; M.U. 91H]

Soln. First of all, we shall show that if we square an odd integer, the result is always an odd integer. For if m=2n+1, then

$$\mathrm{m}^2 = (2\mathrm{n}+1)^2 = 4\mathrm{n}^2 + 4\mathrm{n} + 1$$

$$= 2(2\mathrm{n}^2 + 2\mathrm{n}) + 1$$

Which is an odd integer $2\mathrm{p}+1$ where $\mathrm{p} = 2\mathrm{n}^2 + 2\mathrm{n}$.

Hence it follows that if m^2 is not an odd integer, then m is not an odd integer, i.e. if m^2 bean even integer then m must be necessarily even.

Now, we are equipped to prove that $\sqrt{2}$ is not a rational number. Suppose on the contrary that $\sqrt{2}$ is a rational number.

Let $\sqrt{2} = \frac{p}{q}$ ...(1)

Where p and q are in their lowest terms.

From (1), $p^2 = 2q^2$ ...(2)

But $2q^2$ is an even integer and therefore p^2 is even.

This implies that p must be even. Let $\mathrm{p} = 2\mathrm{m}$.

Then, from (2), $4\mathrm{m}^2 = 2q^2$ or $q^2 = 2\mathrm{m}^2$.

As before, since q^2 is even, q must be even.

Thus p and q are in their lowest terms. Hence we get a contradiction. Hence our supposition that $\sqrt{2}$ is a rational number is false.

Thus we prove that $\sqrt{2}$ is not a rational number.

EX.2. Prove that there is no rational number whose square is 12.

[P.U. 62H]

Soln. Let us suppose that $\frac{\mathrm{p}}{\mathrm{q}}$ is a rational number in its lowest terms whose square is equal to 12. Then

$$\left(\frac{\mathrm{p}}{\mathrm{q}}\right)^2 = 12 \text{ or } \frac{\mathrm{p}^2}{\mathrm{q}^2} = 12 \text{ i.e } \mathrm{p}^2 = 12\mathrm{q}^2$$

$\therefore$ $\qquad \mathrm{p}^2 - 12\mathrm{q}^2 = 0$ ...(1)

We now choose two consecutive positive integers such that 12 lies between the squares of those integers. Obviously such integers are 3 and 4. Thus

$$3^2 < 12 < 4^2$$

i.e. $\qquad 3^2 < \left(\frac{\mathrm{p}}{\mathrm{q}}\right)^2 < 4^2 \Rightarrow 3 < \frac{\mathrm{p}}{\mathrm{q}} < 4$

$\Rightarrow$ $\qquad 3\mathrm{q} < p < 4q$ ...(2)

Now we consider the identity

$$(12q - 3p)^2 - 12(p - 3q)^2$$

$=9(4q - p)^2 - 12(p - 3q)^2$

$=9(16q^2 - 8pq + p^2) - 12(p^2 - 6pq + 9q^2)$

$=144q^2-72\,pq+9p^2-12p^2+72\,pq-108q^2$

$=36q^2-3p^2=-3(p^2-12q^2)$

=0;because of (1).

Hence $\quad (12q-3p)^2 = 12(p-3q)^2$

$\Rightarrow \quad \left(\frac{12q-3p}{p-3q}\right)^2 = 12.$

Now $\frac{12q-3p}{p-3q}$ is a rational number whose denominator $p-3q < q$. Because of (2).

Thus 12 is the square of a rational whose denominator is $< q$.

But we have supposed that 12 is the square of a rational number $\frac{p}{q}$ in its lowest terms. Hence we get a contradiction.

Hence there is no rational number whose square is 12.

The above two examples are covered by a general sort of theorem which is as follows:

1.12. THEOREM

To prove that the square root of any positive rational number which is not a perfect square is not a rational number. [Bhag.94H]

Let m any positive rational number which is not a prefect square.

We want to prove that $\sqrt{m}$ is not a rational number.

Suppose on the contrary that $\sqrt{m}$ is a rational number and let $\sqrt{m} = \frac{p}{q}$

Where p and q are integers prime to each other i.e. they have no factorc in common.

Then $\quad m = \left(\frac{p}{q}\right)^2 \; or \; m = \frac{p^2}{q^2}$

$\Rightarrow \quad mq^2 - p^2 = 0 \quad \ldots(1)$

We choose two consecutive positive integers k and $k+1$ such that

$$k < \sqrt{m} < k+1 \; i.e. k < \frac{p}{q} < k+1$$

$\Rightarrow \quad kq < p < (k+1)p \quad \ldots(2)$

Now consider the identity

$$(mq-kp)^2 - m(p-kq)^2 = (k^2-m)(p^2-mq^2);$$

On simplification as in the previous Ex.

$$= (k^2 - m)\text{x}(0) = 0;$$

Because of (1)

$\therefore \qquad m = \frac{(mq-kp)^2}{(p-kq)^2}\ i.e. \sqrt{m} = \frac{mq-kp}{p-kq}$

But from (2), $p - kq < q$.

This shows that $\sqrt{m}$ is also equal to a rational number in which the denominator is $< q$ and hence $\frac{mq-kp}{p-kq} \neq \frac{p}{q}$.

This is contrary to our supposition and hence our assumption that $\sqrt{m}$ is a rational number is false.

Therefore it follows that $\sqrt{m}$ is not a rational number.

EXAMPLE 1(A)

1. Prove that there is no rational number whose square is equal to 2. **[P.U. 51H; B.U. 55H, 65H; Bhag.65H; R.U. 65H, 74H]**
2. Prove that no rational number can have its square equal to 3. **[B.U. 53H]**
3. Prove that $\sqrt{5}$ is an irrational number. **[Bhag.91H]**
4. Prove that $\sqrt{8}$ is not rational number. **[Bhag.93H]**
5. Examine whether the set of rational numbers contains a solution of the equation $x^2 - 8 = 0$. **[P.U. 67H]**
6. Prove that there is no rational number whose square is 12. **[P.U. 62H]**
7. If d is a positive integer but not the square integer, show that d is not the square of a rational number. **[P.U. 53H; R.U. 70H]**
8. Show that no rational number exists whose nth power is equal to $\frac{a}{b}$ where $\frac{a}{b}$ is a positive fraction in its lowest terms, unless a and b are prefect nth powers. **[P.U. 59H; M.U. 63H; B.U. 69H]**

   [Soln. Let $\frac{p}{q}$ be a rational number in its lowest terms whose nth powers is $\frac{a}{b}$

   i.e. $\left(\frac{p}{q}\right)^n = \frac{a}{b}\ i.e. bp^n = aq^n$ ...(1)

   This equation implies that since p is prime to $q. \therefore p^n$must be a divisor of a.

   Let $a = \lambda p^n$ where λ is an integer. Putting in (1), we get

$$bp^n = \lambda p^n q^n \Rightarrow b = \lambda q^n.$$

Thus, we have $a = \lambda p^n$ and $b = \lambda q^n$.

But a is prime to b, therefore $\lambda = 1$. Thus $a = p^n$ and $b = q^n$ i.e. a and b are perfect nth powers.

9. Prove generally that a rational fraction $\frac{p}{q}$ in its lowest terms cannot be the cube of a rational number unless p and q are both perfect cubes.

[P.U. 57H, 64H]

1.13. QUITE MOTIVATION :

See-the two examples on page 10 amply exhibit to us that there are certain numbers which are not rational numbers. If we accommodate these numbers in our fold then the solution of such equations as $x^2 = 2$ becomes feasible. Hence the question before us is to provide a ground for the induction of these non-rational numbers in our number system. We will read in this chapter how Dedekind provided an analytical method for the induction of such non-rational numbers through the sections of rational numbers. The motivation is provided by the following considerations.

We know that the set of rational numbers Q forms an ordered field. Now since the rational number system is an ordered field, we may imagine the rational numbers to be arranged in order on a line from left to right. If we make a cut of the line in the physical sense at any point α on it, then the set of rational numbers Q is separated into two non-empty proper subsets of it say $L$and R, all rational numbers to the left of α lie in L and all those to the right of α lie in R. Now there are two possibilities: either α corresponds to a rational number or α does not correspond to a rational number (potenuse of a right-angled triangle whose other sides are 1 each= $\sqrt{2}$. If α does not correspond to a rational number, then no element of Q escapes classification. But in order to avoid this short of ambiguity we make the rule that whenever corresponds to a rational number, α will be placed in R only.

Thus whether α corresponds to a rational number or it does not corresponds to a rational number, a cut at the point α of the line separates Q into two non-empty proper subsets L and R, usually called the lower class and upper class respectively such that

(i) $\mathrm{L} \cap \mathrm{R} = \emptyset$

(ii) $\mathrm{L} \cup \mathrm{R} = \mathrm{Q}$
(iii) Each $\mathrm{x} \in \mathrm{L} < each\ y \ \in R$

Because of (i) and (ii) we may write $\mathrm{R} = \mathrm{Q} - \mathrm{L}$ which is therefore the complement of L. The ordered pair (L, R) of non-empty proper subsets of Q is called a Dedekind cut or Dedekind section after the name of Dedekind who first conceived of it. However, since every point is in either L orR it is clear that instead of describing a cut by considering both Land R, we might describe it by specifying say L and then R would be automatically determined as the rational points not in L. Moreover it would be convenient to assume that L has no greatest member. We shall give in the next section a formal definition of a Dedekind cut, independent of any geometrical intuition that is inherent in the above description.

DEDEKIND'S CUT

1.14. Let the set of rational numbers be denoted by the symbol Q. We divide the set of rational numbers into two classes L and R such that
[D1]: $\mathrm{L} \neq \emptyset, \mathrm{R} \neq \emptyset$, This means that there is at least one rational point in L as well as inR.
[D2]: $\mathrm{L} \cup \mathrm{R} = \mathrm{Q}$. This means that there is no rational number which escapes classification. This means that every rational number must find a place either in Lor R.
[D3]: Every rational number belonging to L is less that every rational number belonging to
Ri.e.$x \in L, y \in R \Rightarrow x < y$,

Therefore Land R are called respectively the lower class and the upper class. Thus if a rational number $p \in L$, then $q < p$ then $p$also belongs to L, for if q does not belong to L i.e. if $q \in R$, then p becomes$< q$ which is contrary to our supposition. Similarly if a rational number $r \in R$, then every rational number$>$ belongs to R.

Evidently $[\mathrm{D3}] \Rightarrow \mathrm{L} \cap \mathrm{R} = \emptyset$.

Thus if a rational number$\in L$, then it does not belong to R or if a rational number$\in R$, then it does not belong to L.

Since $\mathrm{L} \cap \mathrm{R} = \emptyset$and $\mathrm{L} \cup \mathrm{R} = \mathrm{Q}$, therefore L and R are complementary sets to each other and we can write $R = Q - L$ (Comp. L').

Such a pair as (L, R) of subsets of Q which satisfies the three conditions stated above is called a section of rational numbers and is denoted by (L, R). As it was stated earlier, L is called the lower class and R the upper class of the section.

Cor. (i) $If\ p \in L\ and\ q < p, then\ q \in L. Let\ q \in R\ Then\ q > p$, according to [D3].

(ii) Similarly it can be shown that if $p \in R\ and\ q > p, then\ q \in R$.

Note: We shall quite often use the symbol $(x_1, x_2)\ or\ (y_1, y_2)$ for a cut.

Ex.1. Let $\boldsymbol{x_1} \equiv \boldsymbol{the\ set\ of\ all\ rational\ numbers} < 2$

And $\boldsymbol{x_2} \equiv \boldsymbol{the\ set\ of\ all\ rational\ numbers} < 2$.

Then (x_1, x_2) is not a cut since the rational number 2 occurs neither in x_1 nor in x_2 so that [D1] is not satisfied.

Ex.2. Let $\boldsymbol{x_1} = \{\boldsymbol{x}, \boldsymbol{x} \in \boldsymbol{Q} | \boldsymbol{x^2} < 2\}$

$\boldsymbol{x_2} = \{\boldsymbol{x}, \boldsymbol{x} \in \boldsymbol{Q} | \boldsymbol{x^2} > 2\}$

Here $x^2 < 2 \Rightarrow -\sqrt{2} < x < +\sqrt{2}$

And $x^2 > 2 \Rightarrow x > \sqrt{2}\ or\ x < -\sqrt{2}$

Thus according to the mode of division, -1 and 1 both $\in x_1$ while -3 and +3 both $\in x_2$ and so condition [D3] is not satisfied.

Hence the division is not a cut.

1.15. TYPES OF SECTIONS OR CUT

When the division or partition of rational numbers has been made in the manner as above, we may have the following types of sections.

(i) The lower class L has a greatest number and the upper class R has no smallest member.

Example, consider $L = \{x \in Q : x \leq 5\}, R = \{x \in Q : x > 5\}$.

In language this means that L is the class of all rational numbers x such that $x \leq 5$ and R is the class of all rational number x such that $x > 5$. It is obvious that the greatest member of L is 5 but R has no smallest member. To show this we take on the contrary any rational member greater than 5, say $5 + \lambda$ whereλ is arbitrarily small, as the smallest member of R.

But we know that there are infinity of rational points between 5 and $5 + \lambda$ howeverλ is infinitesimally small positive number. Hence we cannot assert that a particular rational numberβ, is the smallest number of R such that all the numbers in R are greater thanβ. Thus the upper class R has no smallest rational member.

(ii) The lower class L has no greatest member and the upper class R has the smallest member.

For example,consider $L = \{x \in Q : x < 5\}, R = \{x \in Q : x \geq 5\}$.

That is, L is the class of all rational numbers x such that $x < 5$ and R is the class of all rational numbers x such that $x \geq 5$.

Here L has no greatest member as per discussion above but Rhas the smallest member 5.

(iii) The lower class L has no greatest member and the class has no smallest member.

For example, consider

$$L = \{x \in Q | x < 0, \text{the number } 0 \text{ and all those positive } x \text{ s.t.} x^2 < 2\};$$

$$R = \{x \in Q | x^2 > 2, x > 0\}$$

We would first of all like to verify that this classification represents a section in accordance with the three conditions [D1] to [D3].

Condition no. [D1] and condition no [D3] are obvious. To ensure that condition [D2] is also satisfied we need to show that there is no rational number which escapes classification, For that, we need to show that there is no rational number x such that $x^2 = 2$.

But we have shown in Art 1.10 Ex. 1 that 2 is not a rational number i.e. there is no rational number whose square is 2.

Thus condition [D2] is also satisfied.

Thus we have verified that all the three conditions [D1], [D2], [D3] are satisfied in the case of this classification and hence it represents a section.

Now it remains to be shown that L has no greatest member and R has no smallest member.
If possible, let us suppose that k is the greatest member of L. Then we have $k > 0$ and $k^2 < 2$.
Now, consider the positive number

We have, $2 - \left(\frac{4+3k}{3+2k}\right)^2 = \frac{2(3+2k)^2-(4+3k)^2}{(3+2k)^2}$

$$= \frac{2(9+12k+4k^2)-(16+24k+9k^2)}{(3+2k)^2}$$

$$= \frac{2-k^2}{(3+2k)^2} > 0; since\ k^2 < 2.$$

Hence $\left[\frac{4+3k}{3+2k}\right]^2 < 2.$

Therefore $\frac{4+3k}{3+2k} \in L.$

Also $\frac{4+3k}{3+2k} - k = \frac{4+3k-3k-2k^2}{3+2k} = \frac{2(2-k^2)}{3+2k} > 0$

Therefore $\frac{4+3k}{3+2k} > k.$

Thus there is a positive number $\frac{4+3k}{3+2k}$in L which is greater than k and hence k ceases to be the greatest member of L. So we get a contradiction and therefore the lower class L has no greatest member.
Similarly it can be shown that if we assume that k is the least member of R so that $k^2 > 2$, then it can be shown exactly as before that$\frac{4+3k}{3+2k}$ is still a smaller member of R and therefore we get a contradiction. Hence the upper class R has no least member.

(iv) The classification is inadmissible. To see this, letα be the greatest member of L andβ be the least member of R so that $\alpha < \beta$. Then$\frac{\alpha+\beta}{2}$ will be a positive rational number lying betweenα andβ and so could belong neither to L nor to R and this contradicts our condition [D2] that every rational number belongs to one class or the other.

Thus we see that a section L, R will be only one of the following types-
(i) L has no greatest member but R has a least;
(ii) Lhas a greatest member but R has no least;
(iii) Lhas no greatest member and R has no least.

In the first two cases we say that the section corresponds to a rational number $\alpha = 5$ because the section is generated by the rational number 5.

In the first case, the greatest member of L is 5 and in the second case, the least member of R is 5.

Thus whenever the section (L, R) of rationalsbe such that either the lower class L has the greatest member or the upper class has the least member we say that the section corresponds to a rational number. And such sections or cuts are called rational cuts.

In the third case, the section (L, R) is generated by non-rational number and is such that neither the lower class L has the greatest member nor the upper class R has the least member and therefore we say that the section corresponds to an irrational number. Such cuts are called irrational cuts.

1.16. REAL RATIONAL NUMBERS AND IRRATIONAL NUMBERS

The section (L, R) of rationals which corresponds to a rational number is called a real rational number and the section (L, R) of rationals which does not correspond to a rational number is called a real irrational number.

Henceforth a Dedekind cut shall also be referred as a real number (whether rational or irrational) and the collection of all cuts will be called the system of real numbers.

We shall denote the system of real numbers by the symbol R.

Note: We have stated above that if a section(L, R) of rationals be such that when either (i) L has a greatest member and Rhas no least or (ii) L has no greatest member and R has the least member; then the section corresponds to a rational number. In order to avoid ambiguity, we will only say that if the section(L, R) of rationals be such that L has no greatest member, and R has the least, then the section will correspond to a rational number. The advantage of this adoption will be the following.

First, if L has no greatest member and R has the least, then the section will correspond to a rational number; and secondly if L has no greatest member, so that only one sentence viz. L has no greatest member will serve the purpose of both cases. Consequently we have the following alternative definition of Dedekind's cut which will often be used for simplicity and convenience.

1.17. NEED BASED MODIFIED DEFINITION OF DEDEKIND'S CUT

A Dedekind's cut (or simply a cut) is an ordered pair of sets of rational numbers having the following properties:

[D1]: $x_1 \neq \emptyset, x_2 \neq \emptyset$

[D2]: $x_1 \cup x_2 = Q$.

[D3]: Every rational number in x_1 is less than every rational number in x_2

[D4]: x_1 does not possess a greatest rational number.

Quite analoguously, another definition of Dedekind's cut has been advanced as follows:

A subsetα of rational numbers is said to be a cut if

(i) $\alpha \neq \emptyset$ And also $\alpha \neq Q$

(ii) $\text{If p} \in \alpha \text{ and q} < p \text{ (q rational), then q} \in \alpha$

(iii) α Contains no largest rational. Thus if $\text{p} \in \alpha$, there exists $\text{q} \in \alpha \text{ s. t. q} > p$.

Note: Since α does not carry the sense of a cut, therefore we shall adhere to the modified definition above, although it is quite obvious that $x_1 = \alpha$ and $x_2 = \alpha'$ (complement of α). Any

1.18. THEOREM

Let $(\boldsymbol{x_1}, \boldsymbol{x_2})$**be any section of rational numbers and** $\boldsymbol{\varepsilon} > \boldsymbol{0}$ **be any given rational number number. Then there exist** $\boldsymbol{x} \in \boldsymbol{x_1}$ **and** $\boldsymbol{y} \in \boldsymbol{x_2}$ **such that** $\boldsymbol{y} - \boldsymbol{x} = \boldsymbol{\varepsilon}$.

Proof:**Let** $\boldsymbol{a} \in \boldsymbol{x_1}$**and** $\boldsymbol{b} \in \boldsymbol{x_2}$**.**

We know that the set of rational number is Archimedean, therefore there exists a positive integer n such that

$$nk > b - a \text{ where } k \text{ is a rational number}.$$

This⇒ $a + nk > b$.

Consider the set of rational numbers

$$a, a + k, a + 2k, \dots a + nk$$

Now $a \in x_1 and\ a + nk(> b) \in x_2$.

Therefore there exist two consecutive members of the set (1), say

$$a + rk, a + (r+1)k$$

such that $\quad a + rk \in x_1, a + (r+1)k \in x_2.$

Writing $x = a + rk \; and \; y = a + (r+1)k \; the \; result \; follows.$

1.19. THEOREM

Let (x_1, x_2) **Any section of rational numbers and let $k > 1$ be any given rational number number. Then there exist $x \in x_1$ and $y \in x_2$ such that $\frac{y}{x} = k$.**

Proof: **Let $a \in x_1$ and $b \in x_2$.**

Since $k > 1$, we can write $k = 1 + l, l > 0$.

Now k is given to be a rational number, therefore l is a rational number and consequently al is a rational number. We know that the set of rational number is Archimedean, therefore there exists a positive integer n such that

$$nal > b - a \Rightarrow a + nal > b$$

$$\Rightarrow a(1 + nl) > b.$$

Consider the set of rational numbers

$$a, ak, ak^2, \dots ak^n$$

Since $\quad ak^n = a(1+l)^n > a(1+nl) > b$

$\therefore \quad ak^n \in x_2.$

Thus we find that $a \in x_1 \; and \; ak^n \in x_2.$

This ⇒ that there exists two consecutive members, say ak^r, ak^{r+1} of the set (1) such that

$ak^r \in x_1 \; and \; ak^{r+1} \in x_2.$

Writing $x = ak^r \; and \; y = ak^{r+1} \; we \; find \; \frac{y}{x} = k \; and \; the \; result \; is \; proved.$

1.20. THE ZERO CUT

The cut corresponding to the rational number 0 is called a zero cut. This we shall always denote by the symbol $0^* = (0_1, 0_2)$. Thus $0_1 \equiv$ the set consisting of all rational numbers ≤ 0 and $O_2 \equiv$ the set consisting of all rational numbers > 0

1.21. POSITIVE AND NEGATIVE CUTS

We say that a cut (x_1, x_2) is positive if $x_1, x_2 > (O_1, O_2)$ and negative if $x_1, x_2 < (O_1, O_2)$.

1.22. EQUALITY OF TWO CUTS

Let $\alpha = (x_1, x_2)\ and\ \beta = (y_1, y_2)\ be\ cuts$.

We say $\alpha = \beta$ if every member of x_1 is a member of and y_1 every member of y_1 is a member of x_1, i.e. if $x_1 = y_1$.

Obviously, $x_1 = y_1 \Rightarrow x_2 = y_2$ and conversely. Thus in order to establish the equality of two cuts, it suffices to show that their lower classes are identical or their upper classes are identical.

1.23. SUM OF TWO CUTS [M.U. 90H, 92H]

In order to define the sum of two cuts $\alpha = (x_1, x_2), \beta = (y_1, y_2)$ we consider the following two classes:

(i) The class z_1 consisting of all rational numbers z_1 of the form $z_1 = x_1 + y_1$ where x_1 denotes any member of x_1 and y_1 any member of y_1.

(ii) The class z_2 consisting of all other rational numbers.

We shall first verify that the ordered pair $y = (z_1, z_2)$ is in fact a cut.

(i) Clearly $y = (z_1, z_2)$ is not contain every rational.

(ii) Next we show that z_1 does not contain every rational.

Let $u \in x_2$and $v \in y_2$; u and v are rationals.

(iii) Suppose $u \in z_1$and $t < u$. It is to be shown that t also

Since $u \in z_1$, therefore $u = p + q$ for some $p \in x_1$ and $q \in y_1$.

Choose a rational k such that $t = k + q$. Then since $t < u$,therefore $k < p$. Hence $k \in x_1$.

Thus we see that $t = k + q$ where $k \in x_1$and $q \in y_1$.

Hence $t \in z_1$.

(iv) Suppose $r \in z_1$.

Then $r = p$ for some $p \in x_1$and $q \in y_1$.

Since (x_1, x_2) is a cut, therefore there is a rational $u > p$ such that $u \in x_1$.

Hence $u + q \in z_1$, since $u \in x_1$ and $q \in y_1$.

Also, $u + q > p + q = r$ so that r is not the largest rational inz_1.

Hencez_1 does not contain the largest rational.

Thus we see that all the three conditions are satisfied and hence$y = (z_1, z_2)$ is a cut. This cut is called the sum of two cuts$(x_1, x_2) + (y_1, y_2)$ and we have$(z_1, z_2) = (x_1, x_2) + (y_1, y_2)$. i.e. $\gamma = \alpha + \beta$.

1.24. THE NEGATIVE OF A CUT

Let $\alpha = (X_1, X_2)$ be any cut. Form two classesY_1 andY_2 such thatY_1 consists of all rational numbersy_1 of the form$y_1 < -x_2$ wherex_2 is some member ofX_2 (except whenx_2 is the least member ofX_2) andy_2 consists of all rational membersy_2 of the form$y_2 \geq -x_1$
Wherex_1 is a member ofX_1 together with the negative of the greatest member ofX_1 if such a member exists.
In other words,Y_1 is the set of all rationalsx_2 such that$-x_2$ is an upper bound ofY_1 but not the smallest upper bound.
We have to verify that$[Y_1, Y_2]$ is a cut. For this, we need to verify that $[Y_1, Y_2]$satisfies the three conditions of a cut.

(i) Since$[X_1, X_2]$ is a cut,x_2 is not the empty set and therefore$[Y_1, Y_2]$ is not an empty set.

Moreover,(X_1, X_2) is not the empty set and hence there exists$a \in X_1$.Then$-a \notin Y_1$, since otherwise we would have$-a < -x_2$ for some$x_2 \in X_2$ and this would imply that$x_2 < a$. Thus we get a contradiction that every element ofX_2 is greater than every element of X_1.

Since – $a \notin Y_1, Y_1 \neq$ Q.

(ii) Let$p \in Y_1$ and$q < p$ (a rational). We want to show that$q \in Y_1$.

Since$p \in Y_1$, therefore$-p \notin X_1$ as in case (i). i.e. $-p \in X_2$ and$-q > -p$.

But $\quad Y_1 = \{y_1 | y_1 < -p$ for some$p \in X_2\}$.

Which we can write as

$$Y_1 = \{y_1 | y_1 > p \text{for some } p \in X_2\}$$

Which further means that

$$Y_1 = \{y_1 | y_1 < -p \text{for some } -p \in X_2\}.$$

This in conjunction with the inequality $-q > -p$ will mean that $q \in Y_1$.

(iii)If $y \in Y_1$, then $y < -p$ for some element p of X_2.

Now, $y < \frac{y-p}{2} < -p$ and therefore $\frac{y-p}{2} \in Y_1$.

This shows that there is no largest element of Y_1.

Thus we have shown that $[Y_1, Y_2]$ is a cut which is called the negative or additive inverse of (X_1, X_2) and is denoted by the symbol $-(X_1, X_2)$ i.e. by $-\alpha$.

1.25. EXISTENCE OF ADDITIVE INVERSE

Theorem: Let (X_1, X_2) be any cut. To prove that

$$(\boldsymbol{X_1}, \boldsymbol{X_2}) + [-(\boldsymbol{X_1}, \boldsymbol{X_2})] = (\mathbf{O_1}, \mathbf{O_2}).$$

Proof: Let us define, as before

$$Y_1 = \{y : y \in Q | y < -p \text{for some } p \in X_2\}.$$

We have proved in the preceding theorem that (Y_1, Y_2) is a cut.

We want to prove that $(X_1, X_2) + (Y_1, Y_2) = (\mathrm{O}_1, \mathrm{O}_2)$.

For this, suppose that $r \in X_1 + Y_1$.

Then we can write $r = p + q$ where $p \in X_1$ and $q \in Y_1$.

Since $q \in Y_1 \quad \therefore\ -q \notin X_1$ i.e. $-q \in X_2$

Now $(p \in X_1\ and - q \in X_2) \Rightarrow -q > p \Rightarrow p + q < 0.$

$$\Rightarrow r < 0 \Rightarrow r \in \mathrm{O}_1.$$

Thus we find that $r \in \mathrm{X}_1 + \mathrm{Y}_1 \Rightarrow r \in \mathrm{O}_1$ …(1)

Again, suppose $r \in \mathrm{O}_1$. Then $r < 0$.

We know (theorem 1.17) that there are rationals $p \in \mathrm{X}_1, q \in \mathrm{X}_2$ such that $q - p = -r$.

Now $\quad q \in \mathrm{X}_2 \Rightarrow -q \in \mathrm{Y}_1$.

$\therefore \quad$ From $-r = q - p$, we get $r = p - q$ where $p \in \mathrm{X}_1$ and $-q \in \mathrm{Y}_1$.

Hence $\quad r \in \mathrm{X}_1 + \mathrm{Y}_1$.

Thus we find that $r \in \mathrm{O}_1 \Rightarrow r \in \mathrm{X}_1 + \mathrm{Y}_1$ ...(2)

Combining (1) and (2), we get $\mathrm{X}_1 + \mathrm{Y}_1 = \mathrm{O}_1$.

This completes the proof.

1.26. SUBTRACTION

We define the difference $(X_1, X_2) - (Y_1, Y_2)$ of two cuts(X_1, X_2) and(Y_1, Y_2) by the equation

$(X_1, X_2) - (Y_1, Y_2) = (X_1, X_2) + \{-(Y_1, Y_2)\}$

And so the idea of subtraction is included in the idea of addition.

1.27. EXISTENCE OF ZERO ELEMENT

Theorem :$\boldsymbol{If\ (X_1, X_2)\ be\ any\ cut, then}$

$$(\boldsymbol{X_1}, \boldsymbol{X_2}) + (\mathbf{O_1}, \mathbf{O_2}) = (\boldsymbol{X_1}, \boldsymbol{X_2}).$$

$\boldsymbol{Proof}$ **: Let** $(\boldsymbol{Z_1}, \boldsymbol{Z_2}) = (\boldsymbol{X_1}, \boldsymbol{X_2}) + (\mathbf{O_1}, \mathbf{O_2})$

Let $r \in Z_1$.

Then $r = p + q$ where $p \in X_1$ and $q \in \mathrm{O}_1$ i.e.

Now $p + q < p$, since $q < 0$.

But $p \in X_1 \therefore p + q < X_1$ i.e. $r \in X_1$.

Thus we find $r \in Z_1 \Rightarrow r \in X_1$ ...(1)

Again, let $r \in X_1$. Choose $s > r, s$ rational such that $s \in X_1$.

Put $q = r - s$. Then $q < 0$. $\therefore q \in \mathrm{O}_1$.

From $q = r - s$, we have $r = s + q$

Where $s \in X_1$and $q \in \mathrm{O}_1$.

Hence $r \in Z_1$.

Thus we find that $r \in X_1$ ...(2)

Combining (1) and (2), we get $Z_1 \in X_1$.

Hence X_1 is identical with the lower class $(X_1, X_2) + (\mathrm{O}_1, \mathrm{O}_2)$ and we conclude $(X_1, X_2) + (\mathrm{O}_1, \mathrm{O}_2) + (X_1, X_2)$

1.28. COMMUTATIVE LAW OF ADDITION

Theorem : $\boldsymbol{Let\ \alpha, \beta\ be\ any\ two\ cuts. Then\ \alpha+\beta=\beta+\alpha.}$

[R.U. 69H; M.U. 72H, 90H; Bhag. 74H]

Proof : Let $\alpha=(X_1,X_2)\ and\ \beta=(Y_1,Y_2)$ be any two cuts.

Let $(\mathrm{U}_1,\mathrm{U}_2)=(X_1,X_2)+(Y_1,Y_2)$

And $(\mathrm{V}_1,\mathrm{V}_2)=(Y_1,Y_2)+(X_1,X_2)$.

Now, according to the definition, the lower classU_1 consists of all rational numbers of the formx_1+y_1 wherex_1 andy_1 belong toX_1 andX_2 respectively.

Similarly lower class V_1 is the set of all rational numbers of the form x_1+y_1 with $y_1\in Y_1$and $x_1\in X_1$.

Since $x_1+y_1=y_1+x_1$; for the rational numbers are commutative, it follows that the setU_1 andV_1 are identical and hence $(\mathrm{U}_1,\mathrm{U}_2)=(\mathrm{V}_1,\mathrm{V}_2)$.

That is, $(X_1,X_2)+(Y_1,Y_2)=(Y_1,Y_2)+(X_1,X_2)$

i.e. $\alpha+\beta=\beta+\alpha$.

1.29. ASSOCIATIVE LAW OF ADDITION

Theorem :Let $\boldsymbol{\alpha,\beta,\gamma}$ be any cuts. Then $\boldsymbol{(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)}$

[M.U. 90H]

Proof : Let $\alpha=(X_1,X_2),\beta=(Y_1,Y_2),\gamma=(Z_1,Z_2)$ be any three cuts.

Let $(\mathrm{U}_1,\mathrm{U}_2)=\{(X_1,X_2)+(Y_1,Y_2)\}+(Z_1,Z_2)$

And $(\mathrm{V}_1,\mathrm{V}_2)=(X_1,X_2)+\{(Y_1,Y_2)+(Z_1,Z_2)\}$.

Then as per definition, any member ofU_1 is of the form$(x_1+y_1)+z_1$ and that ofV_1 is of the form$x_1+(y_1++z_1)$where$x_1\in X_1,y_1\in Y_1$and$z_1\in Z_1$.

Since$(x_1+y_1)+z_1=x_1+(y_1++z_1)$ because the associative law holds for rational numbers, therefore$\mathrm{U}_1=\mathrm{V}_1$ which means that

$\{(X_1,X_2)+(Y_1,Y_2)\}+(Z_1,Z_2)=(X_1,X_2)+\{(Y_1,Y_2)+(Z_1,Z_2)\}$

i.e. $\boldsymbol{(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma).}$

1.30. MULTIPLICATION OF CUTS

I. Product of two cuts

Definition- $Let(X_1,X_2)$ and (Y_1,Y_2) be two non-negative cuts. In order to define the product(Z_1,Z_2) of(X_1,X_2) and(Y_1,Y_2) we form the classZ_1 consisting of all rational numbers of the form$z_1=x_1y_1$ where$x_1\in X_1$ and$y_1\in Y_1$ and the classZ_2 consisting of all other rational numbers.

It is easy to verify that the ordered pair(Z_1, Z_2) is actually a cut called the product of two cuts(X_1, X_2) and(Y_1, Y_2) and we write

$$(Z_1, Z_2) = (X_1, X_2)(Y_1, Y_2).$$

The definition of the product can easily be extended to negative cuts.

If(X_1, X_2) is negative and(Y_1, Y_2) is non-negative, then their product is defined as

$$(X_1, X_2)(Y_1, Y_2) = -[\{-(X_1, X_2)\}(Y_1, Y_2)].$$

If(X_1, X_2) and(Y_1, Y_2) are both negative, then we define their product as

$$(X_1, X_2)(Y_1, Y_2) = -[-(X_1, X_2)][-(Y_1, Y_2)].$$

II. **The unit cut**

The cut corresponding to the rational number 1 is called the unit cut. We shall denote it by the symbol (I_1, I_2). Thus

l_1 =the set of all rational numbers< 1 and I_2 consists of all other rational numbers.

Theorem : To prove that for any cut $(\boldsymbol{X_1}, \boldsymbol{X_2})$,

$$(\boldsymbol{X_1}, \boldsymbol{X_2})(\boldsymbol{I_1}, \boldsymbol{I_2}) = (\boldsymbol{X_1}, \boldsymbol{X_2})$$

Proof : Let $(Z_1, Z_2) = (X_1, X_2)(I_1, I_2)$**.**

Let $\qquad\qquad z_1 \in Z_1$.

Then there exist numbers$x_1 \in X_1$ and$e_1 \in I_1$ such that$z_1 = x_1 e_1$. However since $e_1 < 1$, it follows that$z_1 < x_1$ and consequently$z_1 \in X_1$.

Thus we find that $z_1 \in Z_1 \Rightarrow z_1 \in X_1$ …(1)

Conversely, let$x_1{}' \in X_1$. Then sinceX_1 does not possess the greatest member, there exists$x_1{}''$ in X_1 such that$x_1{}'' > x_1{}'$, so that$\frac{x_1{}'}{x_1{}''} < 1$ and consequently belongs toI_1.

Writing $\quad x_1{}' = x_1{}'' \left(\frac{x_1{}'}{x_1{}''}\right)$

We see that $x_1{}'$ has been expressed as the product of a member of X_1 and a member of I_1 Hence $x_1{}' \in Z_1$.

Thus $\quad x_1{}' \in X_1 \Rightarrow x_1{}' \in Z_1 \quad$ ...(2)

Combining (1) and (2), we get $Z_1 = X_1$ and hence the result follows:

III. Commutative law of multiplication

Let $\boldsymbol{\alpha} = (\boldsymbol{X_1}, \boldsymbol{X_2})$ and $\boldsymbol{\beta} = (\boldsymbol{Y_1}, \boldsymbol{Y_2})$ be any two cuts. Then $(\boldsymbol{X_1}, \boldsymbol{X_2})(\boldsymbol{Y_1}, \boldsymbol{Y_2}) = (\boldsymbol{Y_1}, \boldsymbol{Y_2})(\boldsymbol{X_1}, \boldsymbol{X_2})$ i.e. $\boldsymbol{\alpha\beta} = \boldsymbol{\beta\alpha}$.

Proof : Let $\quad (\mathrm{U}_1, \mathrm{U}_2) = (X_1, X_2) + (Y_1, Y_2)$

And $\quad (\mathrm{V}_1, \mathrm{V}_2) = (Y_1, Y_2) + (X_1, X_2)$.

Let $u_1 \in U_1$ and $\mathrm{v}_1 \in \mathrm{V}_1$.

Then u_1 is of the form $x_1 y_1$ where $x_1 \in X_1$ and $y_1 \in Y_1$.

v_1 is of the form $x_1 y_1$ where " " "

Since the commutative law holds for rational numbers, therefore we have $x_1 y_1 = y_1 x_1$.

It follows that $\mathrm{U}_1 = \mathrm{V}_1$ and consequently $(\mathrm{U}_1, \mathrm{U}_2) = (\mathrm{V}_1, \mathrm{V}_2)$.

Hence $(X_1, X_2)(Y_1, Y_2) = (Y_1, Y_2)(X_1, X_2)$.

IV. Associative law of multiplication

***If* $\boldsymbol{\alpha} = (\boldsymbol{X_1}, \boldsymbol{X_2})$, $\boldsymbol{\beta} = (\boldsymbol{Y_1}, \boldsymbol{Y_2})$ and $(\boldsymbol{Z_1}, \boldsymbol{Z_2})$ are any three cuts, then $[(\boldsymbol{X_1}, \boldsymbol{X_2})(\boldsymbol{Y_1}, \boldsymbol{Y_2})](\boldsymbol{Z_1}, \boldsymbol{Z_2}) = (\boldsymbol{X_1}, \boldsymbol{X_2})[(\boldsymbol{Y_1}, \boldsymbol{Y_2})(\boldsymbol{Z_1}, \boldsymbol{Z_2})]$**

i.e. $\quad (\boldsymbol{\alpha\beta})\boldsymbol{\gamma} = \boldsymbol{\alpha}(\boldsymbol{\beta\gamma})$. $\quad$ ...(1)

[M.U. 69H; R.U. 73H]

Proof : Any member of the lower class of the cut on the L.H.S. of (1) is of the form $(x_1 y_1) z_1$ and that of the R.H.S. of (1) is of the form $x_1(y_1 z_1)$ where $x_1 \in X_1, y_1 \in Y_1 and\ z_1 \in Z_1$

Since the associative law holds for rational numbers, we have

$(x_1 y_1) z_1 = x_1(y_1 z_1)$.

Hence the lower classes of both sides of (1) are identical and consequently (1) holds.

V.The distributive laws

Let $\boldsymbol{\alpha} = (\boldsymbol{X_1}, \boldsymbol{X_2})$, $\boldsymbol{\beta} = (\boldsymbol{Y_1}, \boldsymbol{Y_2})$ and $(\boldsymbol{Z_1}, \boldsymbol{Z_2})$ be any three cuts.

Then

(i) $\boldsymbol{\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma}$
(ii) $\boldsymbol{(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma}$**.**

Proof : First we suppose thatα, β, γ are positive cuts, i.e.$\alpha > 0, \beta > 0$ and$\gamma > 0$.

Let $$(\mathrm{U}_1, \mathrm{U}_2) = \alpha(\beta + \gamma)$$

$$= (X_1, X_2)[(Y_1, Y_2)(Z_1, Z_2)]$$

and $$(\mathrm{V}_1, \mathrm{V}_2) = \alpha\beta + \alpha\gamma$$

$$= (X_1, X_2)(Y_1, Y_2) + (X_1, X_2)(Z_1, Z_2).$$

Now, all the negative rational numbers and the number 0 are necessarily members ofU_1andV_1. The positive members of $\mathrm{U}_1$are of the type$x_1(y_1 + z_1)$ and the positive members of $\mathrm{V}_1$are of the type $x_1y_1 + x_1{}'z_1$where$x_1, x_1{}'$ are any positive members inX_1 andy_1z_1 are any positive members of Y_1, Z_1respectively.

Since the distributive law holds for rational numbers, we have

$$x_1(y_1 + z_1) = x_1y_1 + x_1z_1.$$

On taking$x_1{}' = x_1$, we see that every positive membersU_1 is also a member ofV_1.

Again any member$x_1y_1 + x_1{}'z_1$of $\mathrm{V}_1$is clearly a member ofU_1if$x_1{}' = x_1$.

If $x_1 > x_1{}'$ so that $\frac{x_1{}'}{x_1} < 1$, we write

$$x_1{}'z_1 = x_1\left(\frac{x_1{}'}{x_1}\right)z_1 = x_1z_1{}' \text{where } z_1{}' = \frac{x_1{}'}{x_1}z_1.$$

Now $z_1{}'\frac{x_1{}'}{x_1}z < z; since\ \frac{x_1{}'}{x_1} < 1.$

$\therefore$ $z_1{}' \in Z_1$.

Thus $x_1y_1 + x_1'z_1 = x_1y_1 + x_1z_1' = x_1(y_1 + z_1')$

Which clearly belongs to U_1.

In the same way, we can prove that if $x_1 < x_1'$, then every member of V_1 is a member of U_1. Hence $\mathrm{U}_1 = \mathrm{V}_1$ and consequently $\mathrm{U}_2 = \mathrm{V}_2$.

Thus we prove the part (i) of the distributive law viz.

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \; where \; \alpha > 0, \beta > 0, \gamma > 0.$$

The part (ii) can similarly be proved.

Before proceeding to other case, we first prove a lemma viz,

$$= (X_1, X_2)[(Y_1, Y_2) - (Z_1, Z_2)]$$

$$= (X_1, X_2)(Y_1, Y_2) - (X_1, X_2)(Z_1, Z_2).$$

i.e. $\alpha(\beta - \gamma) = \alpha\beta - \alpha\gamma.$ ...(1)

If $\beta = \gamma$, then (1) evidently holds.

If $\beta > \gamma$ then $\beta - \gamma > 0$.

Therefore, we have

$$\alpha\beta = \alpha[\gamma + (\beta - \gamma)]$$

$= \alpha\gamma + \alpha(\beta - \gamma)$; from part (i)

$\Rightarrow \qquad \alpha(\beta - \gamma) = \alpha\beta - \alpha\gamma.$

Thus (1) holds. Similarly it can be shown that (1) holds If $\beta < \gamma$.

We now discuss the main result according as α, β, γ have different signs.

Suppose $\alpha > 0, \beta < 0, \gamma > 0$ and $\beta + \gamma > 0$. Then we have

$$\alpha\gamma = \alpha\{(\beta + \gamma) - \beta\}$$

$= \alpha(\beta + \gamma) - \beta$; according to lemma

$\Rightarrow \qquad \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma.$

Similarly we can discuss other cases. Hence we prove part (i) whether α, β, γ are positive cuts or negative cuts in different orders.

1.31. NON JOINT-RECIPROCAL OF A CUT

Definition-Let (X_1, X_2)be any positive cut. Then its reciprocal or multiplicative inverse is defined to be the cut(Y_1, Y_2)such that

(i) The lower classY_1 contains all negative rational numbers, the numbers, the number zero and the reciprocals of all rational numbers inX_2 excluding the least member ofX_2 if that exists.

(ii) The upper classY_2 consists of the reciprocals of all positive numbers inX_1 together with the greatest member ofX_1 if there is one.

It can be easily verified that(Y_1, Y_2) is a cut.

We shall denote the reciprocal of(X_1, X_2) be the symbol$(X_1, X_2)^{-1}$

If(X_1, X_2) is negative, then we define its reciprocal to be$-[-(X_1, X_2)]^{-1}$

The reciprocal of the zero cut(O_1, O_2) is not defined.

Theorem :$\boldsymbol{If\ \alpha = (X_1, X_2)\ is\ any\ positive\ cut, then}$

$$\boldsymbol{(X_1, X_2)(X_1, X_2)^{-1} = (I_1, I_2)\ i.e.\ \alpha.\alpha^{-1} = 1.}$$

Proof : Let $(Y_1, Y_2) = (X_1, X_2)^{-1}$.

Then $Y_1 \equiv$ the set of all negative rational numbers, the number zero and the reciprocals of all rational numbers in X_2

$$\equiv Q^{-1} \cup \{y \in Q^+; y = \frac{1}{a}, a \in X_2\}.$$

We shall use the following symbols :

$[X_1]_p$ =the set of all $+ve$ rational in X_1;

$[Y_1]_p$ =the set of all $+ve$ rational in Y_1;

$[I_1]_p$ =the set of all $+ve$ rational in I_1;

Since $O_1 \subset X_1$, therefore $[X_1]_p \neq \emptyset$.

Also, there exists an element $a_1 \in X_1$ such that $a_1 > 0$.

Therefore the set $[Y_1]_p = \{y \in Q^+; x = \frac{1}{a}, a \in X_2\} \neq \emptyset$.

This shows that (Y_1, Y_2) as defined above is a $+ve$ cut.

Now (X_1, X_2) is a $+ve$ cut and (Y_1, Y_2) is a $+ve$ cut implies that $(X_1, X_2)(Y_1, Y_2)$ is a positive cut.

Let $\quad (Z_1, Z_2) = (X_1, X_2)(Y_1, Y_2)$

And let $[Z_1]_p \equiv$ the set of all $+ve$rationalsin Z_1.

We need to prove that $[Z_1]_p = [I_1]_p$

Let $\quad z \in [Z_1]_p$

Then $z = ab$ for some $a \in [X_1]_p$and $b \in [Y_1]_p$.

Also, $\quad b \in [Y_1]_p \Rightarrow b = \frac{1}{a'}$ for $a' \in X_2$.

Now, $\quad a \in [X_1]_p$and $a' \in X_2 \Rightarrow a < a'$.

Therefore $z = ab = a.\frac{1}{a'} < 1 \quad (\because a < a')$

$\therefore \; z \in [I_1]_p$.

Thus $\quad z \in [Z_1]_p \Rightarrow z \in [I_1]_p$

Hence $\quad [Z_1]_p \subset [I_1]_p \qquad \ldots(1)$

Again, let $u \in [I]_p$ so that $0 < u < 1$.

Since $\quad u < 1 \therefore \frac{1}{u} > 1$.

Therefore from the theorem 1.18, there exists an element $a_1 \in [X_1]_p$

Such that $\frac{a_1}{u} \in X_2$.

But $\quad \frac{a_1}{u} \in X_2 \Rightarrow \frac{u}{a_1} \in [Y_1]_p$.

Thus $\quad u = a_1 \cdot \frac{u}{a_1}$ *where* $a_1 \in [X_1]_p$ *and* $\frac{u}{a_1} \in [Y_1]_p$.

This shows that $u \in [Z_1]_p$.

$\therefore \; u \in [I_1]_p \Rightarrow u \in [Z_1]_p$.

Hence $[I_1]_p \subset [Z_1]_p$. ...(2)

From (1) and (2), we find that $[Z_1]_p = [I_1]_p$ and consequently,

$$(Z_1, Z_2) = (I_1, I_2).$$

Hence the theorem is proved.

Cor. It follows from above that if $\alpha = (X_1, X_2)$ is a negative cut, then there exist a cut $\beta = (Y_1, Y_2)$ such that $\alpha\beta = (I_1, I_2)$; of course β should be a negative cut.

This follows as under.

Let $\alpha = (X_1, X_2)$ be a negative cut, therefore $-(X_1, X_2)$ is a positive cut. Similarly $-(Y_1, Y_2)$ is a $+ve$ cut.

By the preceeding theorem there exists a cut $\beta = -(Y_1, Y_2)$ such that $\alpha\beta$ i.e. $[-(X_1, X_2)][-(Y_1, Y_2)]$

i.e. $(X_1, X_2)(Y_1, Y_2) = (I_1, I_2)$.

Thus we prove that for each non-zero cut α, there exists a cut β such that $\alpha\beta = (I_1, I_2)$.

Note : β is called a multiplicative inverse of α.

Division – We define the division of a cut (X_1, X_2) by a non-zero cut (Y_1, Y_2) by $(X_1, X_2)(Y_1, Y_2)^{-1}$ and denote it by $\frac{(X_1, X_2)}{(Y_1, Y_2)}$.

The division by the zero cut is not defined.

1.32. ORDER RELATION IN CUTS

Let $\alpha = (X_1, X_2)$ and $\beta = (Y_1, Y_2)$ be two cuts.
Then $\alpha < \beta$ if every member of X_1 is a member of Y_1 but every member of Y_1 is not a member of X_1 i.e. if X_1 is a proper subset of Y_1 i.e. $X_1 \subset Y_1$ (or what is the same if is a proper subset of X_2.)
Equivalently, we say that $\beta > \alpha$.

Ex. For any positive cuts $\boldsymbol{\alpha} = (\boldsymbol{X_1}, \boldsymbol{X_2}), \boldsymbol{\beta} = (\boldsymbol{Y_1}, \boldsymbol{Y_2})$, prove that $(\boldsymbol{X_1}, \boldsymbol{X_2}) + (\boldsymbol{Y_1}, \boldsymbol{Y_2})$ is positive i.e. $\boldsymbol{\alpha} > 0, \boldsymbol{\beta} > 0 \Rightarrow \alpha + \beta > 0$.

[M.U. 74H]

Proof :Let $\alpha = (X_1, X_2)$and $\beta = (Y_1, Y_2)$.

Also, let $\gamma = (X_1, X_2) + (Y_1, Y_2) = (Z_1, Z_2)$.

Sinceα is positive, therefore$0_1 \subset X_1$.

Similarly$0_1 \subset Y_1$.

Since$z_1 \in Z_1$ is obtained by adding elements of X_1 with those ofY_1.

We get$0_1 \subset Z_1$ which implies that(Z_1, Z_2) is a positive cut.

Thus $\alpha > 0$ (a positive cut), $\beta > 0 \Rightarrow \alpha + \beta > 0$.

(i) Trichotomy Law of Cuts

From the definition of cuts, it is evident that one and only one of the following relations holds.

(i) X_1 is a proper subset of Y_1
(ii) Y_1 is a proper subset of X_1
(iii) $X_1 = Y_1$.

From this, we conclude that for any two cuts

$$\alpha = (X_1, X_2), \beta = (Y_1, Y_2)$$

We have either

(i) $(X_1, X_2) < (Y_1, Y_2)$ or $(Y_1, Y_2) < (X_1, X_2)$ or $(X_1, X_2) = (Y_1, Y_2)$

i.e. either $\alpha < \beta$ or $\beta < \alpha$ or $\alpha = \beta$.

Thus the trichotomylaw for the order relation in the system of cuts (real numbers) holds.

(ii) Transitivity of the order relation

$\boldsymbol{Let\alpha} = (\boldsymbol{X_1}, \boldsymbol{X_2}), \boldsymbol{\beta} = (\boldsymbol{Y_1}, \boldsymbol{Y_2})$ $\boldsymbol{and}$ $\boldsymbol{\gamma} = (\boldsymbol{Z_1}, \boldsymbol{Z_2})$**be three cuts such that** $\boldsymbol{\alpha} < \beta$ and $\beta < \gamma$ **i.e.** $(\boldsymbol{X_1}, \boldsymbol{X_2}) < (\boldsymbol{Y_1}, \boldsymbol{Y_2})$ **and** $(\boldsymbol{Y_1}, \boldsymbol{Y_2}) < (\boldsymbol{Z_1}, \boldsymbol{Z_2})$**. Then** $(\boldsymbol{X_1}, \boldsymbol{X_2}) < (\boldsymbol{Z_1}, \boldsymbol{Z_2})$ **i.e.** $\boldsymbol{\alpha} < \gamma$**.**

By definition,$X_1 \subset Y_1$and$Y_1 \subset Z_1$.

This$\Rightarrow$ that$Y_1 \subset Z_1$ and hence $(X_1, X_2) < (Z_1, Z_2)$ i.e. $\alpha < \gamma$

Thus we have shown that $\alpha < \gamma$ and $\beta < \gamma \Rightarrow \alpha < \gamma$.

1.33. TWO THEOREMS

Theorem I. Let $\boldsymbol{a} = (\boldsymbol{A_1}, \boldsymbol{A_2}), \boldsymbol{b} = (\boldsymbol{B_1}, \boldsymbol{B_2})$ and $\boldsymbol{c} = (\boldsymbol{C_1}, \boldsymbol{C_2})$ be cuts of rational numbers. Then

$$(\boldsymbol{A_1}, \boldsymbol{A_2}) > (\boldsymbol{B_1}, \boldsymbol{B_2}) \Rightarrow (\boldsymbol{A_1}, \boldsymbol{A_2}) + (\boldsymbol{C_1}, \boldsymbol{C_2}) > (\boldsymbol{B_1}, \boldsymbol{B_2}) + (\boldsymbol{C_1}, \boldsymbol{C_2})$$

$\boldsymbol{i.e.}$ $\quad \boldsymbol{a} > b \Rightarrow a + c > b + c$. **[Bhag. 74H]**

Proof : Let $a = (A_1, A_2), b = (B_1, B_2)$ and $c = (C_1, C_2)$.

Let $\quad (A_1, A_2) + (C_1, C_2) = (U_1, U_2)$

and $\quad (B_1, B_2) + (C_1, C_2) = (V_1, V_2)$.

In order to prove that $(U_1, U_2) > (V_1, V_2)$, we need to prove that $V_1 \subset U_1$.

Now, from the definition,

$$U_1 = \{x_1 + z_1 \in \mathrm{Q} | x_1 \in A_1, z_1 \in C_1\} \quad ...(1)$$

$$V_1 = \{y_1 + z_1 \in \mathrm{Q} | y_1 \in B_1, z_1 \in C_1\} \quad ...(2)$$

It is given that $(A_1, A_2) > (B_1, B_2{}')$ i.e. $B_1 \subset A_1$ i.e. B_1 is a proper subset of A_1. Hence (2) in conjunction with (1) implies there are more rational points in U_1 than in V_1. That is, $V_1 \subset U_1$.

Hence the result follows :

Cor. $a \geq b \Rightarrow a + c \geq b + c$.

Theorem II. Let $\boldsymbol{a} = (\boldsymbol{A_1}, \boldsymbol{A_2}), \boldsymbol{b} = (\boldsymbol{B_1}, \boldsymbol{B_2})$ and $\boldsymbol{c} = (\boldsymbol{C_1}, \boldsymbol{C_2})$ be cuts of rational numbers. Then

$$(\boldsymbol{A_1}, \boldsymbol{A_2}) < (\boldsymbol{B_1}, \boldsymbol{B_2}), (\boldsymbol{C_1}, \boldsymbol{C_2}) > (\mathbf{O_1}, \mathbf{O_2})$$

$$\Rightarrow (\boldsymbol{A_1}, \boldsymbol{A_2})(\boldsymbol{C_1}, \boldsymbol{C_2}) < (\boldsymbol{B_1}, \boldsymbol{B_2})(\boldsymbol{C_1}, \boldsymbol{C_2}).$$

$\boldsymbol{i.e.}$ $\quad \boldsymbol{a} < b, c > 0 \Rightarrow \boldsymbol{ac} < bc$ **[P.U. 71H]**

Proof : First of all we suppose that $a = (A_1, A_2)$ and $b = (B_1, B_2)$ both are positive.

Let $\quad (A_1, A_2)(C_1, C_2) = (U_1, U_2) \quad ...(1)$

and $\quad (B_1, B_2)(C_1, C_2) = (V_1, V_2) \quad ...(2)$

we need to prove that $U_1 \subset V_1$.

we observe first that since $a > 0, b > 0, c > 0$, therefore $ac > 0, bc > 0$

i.e. $(U_1, U_2) > (O_1, O_2)$ and $(V_1, V_2) > (O_1, O_2)$.

It is given that $(A_1, A_2) < (B_1, B_2)$ so that $A_1 \subset B_1$. It means that there are more positive rationals in B_1 than in A_1 and consequently there will be more positive rationals in V_1 than in U_1 and hence $U_1 < V_1$.

The other cases may be dealt with similarly.

1.34. The RATIONAL AND IRRATIONAL CUTS

I. Sum and product of two rational cuts

Let (X_1, X_2) and (Y_1, Y_2) be any two rational cuts corresponding to the rational numbers x and y respectively.

That is, $X_1 \equiv$ the set of all rationals $< x$

$X_2 \equiv$ the set of all rationals $\geq x$.

Similarly $Y_1 \equiv \{r \in Q : r < y\}$

$Y_2 \equiv \{r \in Q : r \geq y\}$.

It is easy to see that $x + y$ and xy are the least members of upper classes of $(X_1, X_2) + (Y_1, Y_2)$ and $(X_1, X_2)(Y_1, Y_2)$ respectively.

Since $x + y$ and xy are rational numbers, therefore the sum and product of rational cuts is a rational cut.

II. Sum and product of a rational an irrational cuts

Let (X_1, X_2) be a rational cut and let (Y_1, Y_2) be any irrational cut. Then $-(X_1, X_2)$ is also rational.

Now consider the identity

$$[(X_1, X_2) + (Y_1, Y_2)] + [-(X_1, X_2)] = (Y_1, Y_2).$$

From the identity it follows that if is rational, then is rational. But it is given that is not rational. Hence cannot be rational. i.e. it must be irrational. In the same manner, by considering the identity

$$\{(X_1, X_2)(Y_1, Y_2)\}(X_1, X_2)^{-1} = (Y_1, Y_2)$$

Where $(X_1, X_2)^{-1} \neq (O_1, O_2)$, We can show that the product$(\mathrm{X}_1, \mathrm{X}_2)(\mathrm{Y}_1, \mathrm{Y}_2)$ of irrational.

Note : The sum and product of irrationalcuts may be rational or irrational.

For example, if $(\mathrm{X}_1, \mathrm{X}_2)$ is irrational, then so is $-(\mathrm{X}_1, \mathrm{X}_2)$.

Now, consider the identity $(\mathrm{X}_1, \mathrm{X}_2) + \{-(\mathrm{X}_1, \mathrm{X}_2)\} = (\mathrm{O}_1, \mathrm{O}_2)$.

Since$(\mathrm{O}_1, \mathrm{O}_2)$ is rational, therefore sum of two irrational cuts may be rational.

Similarly from the identity$(\mathrm{X}_1, \mathrm{X}_2)(\mathrm{X}_1, \mathrm{X}_2)^{-1} = (\mathrm{I}_1, \mathrm{I}_2)$, it follows that the product of two irrational cuts may be rational.

1.35. DENSITY THEOREM

Theorem : Between any two distinct real numbers$\boldsymbol{\alpha}$ and$\boldsymbol{\beta}$ there are infinity of rational numbers.

[B.U. 53H, 68H; M.U.68H, 73H; R.U. 70H; Bhag 67H, 90H;Mithila 78H]

Proof : Let $\alpha = (\mathrm{X}_1, \mathrm{X}_2)$and $\beta = (\mathrm{Y}_1, \mathrm{Y}_2)$ be two cuts and let $\alpha < \beta$ i.e. $\mathrm{X}_1 \subset \mathrm{Y}_1$.

Our purpose will be served if we can show that there lie two rational numbers between $\alpha$and β.

To this end, we wish to show that there exists a rational cut $r = (\mathrm{R}_1, \mathrm{R}_2)$ of the form

$$\mathrm{R}_1 = \{x : x \in \mathrm{Q}, x < r\}$$

such that $\mathrm{X}_1 \subset \mathrm{R}_1 \subset \mathrm{Y}_1$.

Since $\alpha < \beta$, there exists a rational number p such that $p \in \mathrm{Y}_1$ but $p \notin \mathrm{X}_1$.

Since β is a cut, one of the defining properties of a cut asserts that in Y_1.

There is no largest rational number.

Hence there exists $r \in \mathrm{Y}_1$ such that $p < r$.

But $r < r$ is absurd and hence $r \notin \mathrm{R}_1$.

Now $\quad r \in \mathrm{Y}_1$ and $r \notin \mathrm{R}_1 \Rightarrow \mathrm{R}_1 \subset \mathrm{Y}_1$ ...(1)

Again, $\quad p < r \Rightarrow p \in \mathrm{R}_1$.

Given $\quad p \notin \mathrm{X}_1$.

Therefore $p \in \mathrm{R}_1$ and $p \notin \mathrm{X}_1 \Rightarrow \mathrm{X}_1 \subset \mathrm{R}_1$ ...(2)

From (1) and (2), we get

$$\mathrm{X}_1 \subset \mathrm{R}_1 \subset \mathrm{Y}_1.$$

Similar is the case when $\alpha < \beta$.

In a similar manner, we can show that there is another rational cut between $\alpha$and β.

Thus there must be at least two unequal rational numbersr_1 andr_2 say, between $\alpha$and β. But we know that there are infinity of rational numbers between any two unequal rational numbers (here) $r_1$and r_2. Hence there are infinity of rational numbers between two different real numbers $\alpha$and β.

1.36. THEOREM

Between any two different real numbers, there are infinity of irrational numbers.

[B.U. 68H;Bhag 67H, 90H; R.U. 70H; M.U.68H,73H]

Proof : Suppose $\alpha < \beta$.

We have proved in the preceding theorem that there lie an infinite number of rational numbers between $\alpha$and β.

We pick up two rational numbers r_1 and r_2 from this infinite set of rationals such that

$$\alpha < r_1 < r_2 < \beta.$$

We want to prove that there exists an irrational number lying between $r_1$and r_2.

For this, we consider the real number

$$r = r_{1+\frac{r_2-r_1}{\sqrt{2}}}$$

Clearly r is an irrational real number and we find that

$$r > r_1 \text{because } r_2 > r_1;$$

And also $r < r_2$ because

$$r_2 - r = r_2 - r_1 - \frac{r_2-r_1}{\sqrt{2}}$$

$$= (r_2 - r_1)\left(1 - \frac{1}{\sqrt{2}}\right) \text{which is} > 0.$$

Thus $\qquad r_1 < r < r_2.$

Hence $\qquad \alpha < r_1 < r < r_2 < \beta.$

Thus we have shown that there exists one irrational number r between $\alpha$and β. Butr_1 and r_2 are arbitrary rationals lying between $\alpha$and β. Now since there are infinity of rational numbersr_1 and r_2 between $\alpha$and β, hence we get infinity of rational real numbers according to the scheme above. This proves the theorem.

1.37. SECTION OF THE REAL NUMBERS : COMPLETENESS THEOREM

Let us recall that in art 1.13 we considered the sections of rational numbers i.e. we divided the set Qof rational numbers into two classes L and R characterised by the following properties:

(i) $L \neq \emptyset, R \neq \emptyset$i.e. each class is non-empty.

(ii) Every rational number belongs to either L or R i.e. no rational number escapes classification.

(iii) Every member of L is less than every member of R i.e. $x \in L, y \in R \Rightarrow x < y$.

The two classes L and R are called the lower class and upper class respectively.

This type of division of the set Q into two classes is called a 'section'.

The set of all upper bounds of L is the set R and the set of all lower bounds of R is the set L. Sometimes, we use the term 'maximum' or 'greatest' for the least upper bound (lub) of a set when it belongs to the set. When the glb of a set is in the set, it is sometimes called the 'minimum' or 'least' of the set.