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SEVERAL FIXED POINT THEOREMS IN COMPLEX INVOLUTION BANACH SPACES

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INTRODUCTION :

This paper is committed to the find out of several fixed point theorems in Banach spaces. In section 1.1, we have proved various fixed point theorems on concurrence points of certain complex involutions in Banach spaces employing Lipschitzian involution [16], S. Sessa [17] and Khan & Imdad [13] contractive conditions which seem to be a contribution to the existing results and which in turn generalize and unify several other results.

Preliminaries :

Let R_+ be the set of all non-negative reals and H_i be the family of all functions from R_+^i to R_+ for each positive integer i, which are upper semi continuous and non decreasing in each coordinate variable.

Now, the subsequent definitions are borrowed by numerous authors the weak-commutativity condition introduced by Sessa [17] in metric space, which can be described in normed linear space stated as

Key words:- complex involution.concurrence points,weak commutativity

Fixed point theorems of composite involutions in banach spaces :

In this Section, we have obtained some fixed point theorems on coincidence points of certain composite involutions with some new contractive type conditions, which are extension and generalizations of Goebal and Zlotkiewicz [4], Khan-Imdad [13], Iseki [11].

Motivated from the contractive conditions given by Pachpatte [15]. We prove the following result by using this lemma.

Let x be an arbitarary point in K and $A = \frac{1}{2}(T + I)$, Define $y = Ax,\ z = Ty\ and$ « $= 2y - z$, we shall make repeated use of the following equivalent values. Where K stands for closed and convex subset of a Banach space X and $T : K \to K$. Therefore we state the lemma.

Lemma :

$$\|y - Tx\| = \|x - y\| = 1/2\|x - Tx\|$$

and $$\|x - Tx\| = 2\|Ax - x\|, \|y - Ty\| = 2\|A^2x - Ax\|$$

Now we prove the following result.

Theorem :

Let F,G,S and T be self mappings of a Banach space X satisfying

(i) The pair (ST, FG) commute

(ii) The pair (S,T) and (F,G) are composite involution

(iii) $\|STx - STy\|^3 \le h\left(\|FGx - FGy\|.\|FGx - STx\|.\|FGy - STy\|\right)$

....(1.1)

for every $x,y \in X$, where $0 \le h < 2$, then FG and ST have a coincidence point x_0, i.e., $FGx_0 = STx_0$. Moreover, if $h < 1$ and the pairs (S,T), (ST,F), (ST,G), (F,G), (FG,S) and (FG,T) commute at the foregoing fixed point x_0, then x_0 also remains the unique common fixed point of S,T,F and G.

Proof : From (i) and (ii) it follows that $(STFG)^2 = I$. Now using (1.1), we have,

$$\|STFG\,Fx - STFG\,Fy\| \le h^{1/3}\Big(\|(FG)^2 Fx - (FG)^2 Fy\| \,.\, \|(FG)^2 Fx - (STFG)Fx\|.$$

$$\|(FG)^2 Fy - (STFG)Fy\|\Big)^{1/3}$$

if we set $Fx = z$ and $Fy = w$, then we get

$$\|STFG\,z - STFG\,w\| \le h^{1/3}\left(\|z - w\| \;\|z - (STFG)z\|\|w - (STFG)w\|\right)^{1/3}$$

Since the map $STFG$ is an involution, therefore, we define $w = Az$, ₵ $=(STFG)w$ and µ $= 2w$-₵ and note the values given in Lemma1.1.1.

Now consider

$$\|¢ - z\| = \|(STFG)w - (STFG)^2 z\|$$

$$\le h^{1/3}\left(\|w - (STFG)z\| .\|w - (STFG)w\|.\|(STFG)z - (STFG)^2 z\|\right)^{1/3}$$

$$\le h^{1/3}\left(\|w - (STFG)z\|.\|w - (STFG)w\|.\|(STFG)z - z\|\right)^{1/3}$$

$$\le h^{1/3}\left(\frac{1}{2}\|z - (STFG)z\|.\|w - (STFG)w\|.\|z - (STFG)z\|\right)^{1/3}$$

....(1.2)

Similarly, by Lemma 1.1.1,

$$\|« - z\| = \|2w - ¢ - z\| = \|(STFG)z - (STFG)w\|$$

$$\le h^{1/3}\left(\|z - w\|.\|z - (STFG)z\|.\|w - (STFG)w\|\right)^{1/3}$$

$$\le h^{1/3}\left(1/2\|z - (STFG)z\|.\|z - (STFG)z\|.\|w - (STFG)w\|\right)^{1/3}$$

....(1.3)

Thus, by using inequality (1.9) and (1.10), we get

$$\|¢ - «\| = \|¢ - z\| + \|z - «\|$$

$$\leq 2h^{1/3}\left(\frac{1}{2}\|z - (STFG)z\| .\|z - (STFG)z\|.\|w - (STFG)w\|\right)^{1/3}$$

But

$$\|¢ - «\| = 2\|w - (STFG)w\|,$$

so that above inequality yields

$$\|w - (STFG)w\| \leq h^{1/3}\left(\frac{1}{2}\|z - (STFG)z\|.\|z - (STFG)z\|.\|w - (STFG)w\|\right)^{1/3}$$

This implies that

$$\|w - (STFG)w\| \leq (h/2)^{1/2}\|z - (STFG)z\|$$

Making use of Lemma 1.4.1, gives

$$\|A^2 z - Az\| \leq (h/2)^{1/2}\|Az - z\|$$

Consequently, proceeding inductively, we obtain

$$\|A^{n+1} z - A^n z\| \leq (h/2)^{n/2}\|Az - z\|$$

Since $h < 2$, it follows that $\|A^{n+1} z - A^n z\| \to 0$ as $n \to \infty$. Thus $\{A^n x\}$ is a Cauchy sequence and converges, to some point x_0 is X. We obtains, therefore $Ax_0 = x_0$ and so $(STFG)\, x_0 = x_0$.

So $(STFG)$ has at least one fixed point say x_0 in X i.e., $(STFG)\, x_0 = x_0$. Now using $(ST)^2 = I$, we get $FGx_0 = STx_0$ i.e. is a coincidence point of ST and FG.

Now

$$\|STx_0 - x_0\| = \|STx_0 - ST(FGx_0)\|$$

$$\leq h^{1/3}\left(\|FGx_0 - FG(STx_0)\|.\|FGx_0 - STx_0\|.\|FG(STx_0) - ST(STx_0)\|\right)^{1/3}$$

$$\leq h^{1/3}\left(\|STx_0 - x_0\|.0.0\right)^{1/3}$$

$$= 0$$

yielding thereby $STx_0 - x_0 = 0$, or $STx_0 = x_0$ i.e., x_0 is a fixed point of ST and hence of FG.

To prove the uniqueness of common fixed point x_0, let y_0 be another fixed point of ST and FG, then

$$\|x_0 - y_0\| = \|STx_0 - STy_0\|$$
$$\le h^{1/3}\left(\|FGx_0 - FGy_0\|.\|FGx_0 - STx_0\|.\|FGy_0 - STy_0\|\right)^{1/3}$$
$$\le h^{1/3}\left(\|x_0 - y_0\|.0.0\right)^{1/3}$$
$$= 0$$

which implies that $x_0 = y_0$. i.e., x_0 is a unique common fixed point ST and FG.

Now using the commutativity of the pairs (F,G), (S,T), (FG,S), (FG,T), (ST,F), (ST,F) and (ST,G) at x_0 one can write.

$Sx_0 = S(TSx_0) = ST(Sx_0)$, $Fx_0 = F(GFx_0) = FG(Fx_0)$,
$Tx_0 = T(TSx_0) = ST^2x_0 = ST(Tx_0)$, $Gx_0 = G(GFx_0) = FG(Gx_0)$,
$Sx_0 = S(FGx_0) = FG(Sx_0)$, $Fx_0 = F(STx_0) = ST(Fx_0)$,
$Tx_0 = T(FGx_0) = FG(Tx_0)$, $Gx_0 = G(STx_0) = ST(Gx_0)$,

which show that Fx_0, Gx_0, Sx_0 and Tx_0 is a common fixed point of the pair (ST,FG) which due to uniqueness of the common fixed point of the pair (ST,FG) get us.

$x_0 = Sx_0 = Tx_0 = Fx_0 = Gx_0$

This completes the proof.

As the consequences of our Theorem 1.1.2, we get the following result by putting $FG=I$ and $S=I$.

Corallary :

Let T be self mappings of a Banach space X satisfying

(i) $T^2 = I$

(ii) $\|Tx - Ty\|^3 \le h\left(\|x - y\|.\|x - Tx\|.\|y - Ty\|\right)$

for every $x,y \in X$, where $0 \le h < 2$, then T has at least one fixed point.

By unifying several well known contractive conditions in fixed point theory, Delbosco [2] defined a **g – contraction** as follows

$$d(Tx,Ty) \le g(d(x,y), d(x,Tx), d(y,Ty))$$

where g: $R_+^3 \to R_+$ is a continuous function having the properties.

(i) $g(1,1,1) = h < 1$ and

(ii) for $u,v \ge 0$ such that $u \le g(u,v,v)$ *or* $u \le g(v,u,v)$ or $u \le g(v,v,u)$

then $u \le hv$.

However, we shall assume function g to have somewhat different properties from that defined by Delbosco [2]

Let ₵ be the set all real valued contributions function of

g : $R_+^3 \to R_+$ satisfies the condition

(i) $g(1,1,1) = h < 2$

(ii) if $u, v \geq 0$ are such that either $u \leq g(v, 2v, u)$ or $u \leq g(v, u, 2v)$or $u \leq g(u, 2v, v)$, them $u \leq hv$

Now, we prove the following theorem,

Theorem:

Let *F*, *G*, *S* and *T* be self mappings of a Banach space *X* satisfying.

(i) The pair (*ST*,*FG*) commute,

(ii) The pairs (*S*, *T*) and (*F*, *G*) are composite involutions,

(iii) $\|STx - STy\| \leq g\left(\|FGx - FGy\|, \|FGx - STx\|, \|FGy - STy\|\right)$

....(1.4)

for all $x, y \in X, g \in ₵$, then *FG* and *ST* have a coincidence point x_0, i.e. $FGx_0 = STx_0$, Moreover if the pairs (*S*,*T*), (*ST*, *F*), (*ST*, *G*), (*F*, *G*), (*FG*, *S*) and (*FG*, *T*) commute at the foregoing fixed point x_0, then x_0 also remains the unique common fixed point of *S*, *T*, *F* and *G*.

Proof : From (i) and (ii) it follows that $(STFG)^2 = I$. Now using (1.11), we have

$$\|STFG\ Fx - STFG\ Fy\| \leq g\left(\|(FG)^2 Fx - (FG)^2 Fy\|, \|(FG)^2 Fx - (STFG)Fx\|,\right.$$
$$\left.\|(FG)^2 Fy - (STFG)Fy\|\right)$$

If we set $Fx = z$ and $Fy = w$, then we get

$$\|STFG\, z - STFG\, w\| \leq g\left(\|(z - w\|, \|(z - STFG\, z\|, \|(w - STFG\, w\|\right) \quad \text{....(1.5)}$$

Since the map *STFG* is an involution, therefore, we define $w = Az$, $₵ = (STFG)w$ and $« = 2w - ₵$ and note the values given in Lemma 1.4.1.

Now consider

$$\|₵ - z\| = \|(STFG)w - (STFG)^2 z\|$$
$$\leq g\left(\|w - (STFG)z\|, \|w - (STFG)w\|, \|(STFG)z - (STFG)^2 z\|\right)$$
$$\leq g\left(\|w - (STFG)z\|, \|w - (STFG)w\|, \|(STFG)z - z\|\right)$$
$$\leq g\left(\frac{1}{2}\|z - (STFG)z\|, \|w - (STFG)w\|, \|z - (STFG)z\|\right) \qquad \text{....(1.6)}$$

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by Lemma1.4.1

Again

$$\|« - z\| = \|2w - ¢ - z\| = \|(STFG)z - (STFG)w\|$$

$$\leq g\left(\|z - w\|, \|z - (STFG)z\|, \|w - (STFG)w\|\right)$$

$$\leq g\left(\frac{1}{2}\|z - (STFG)z\| \|z - (STFG)z\|, \|w - (STFG)w\|\right) \qquad ....(1.7)$$

But

$$\|¢ - «\| \leq \|¢ - z\| + \|z - «\|$$

And so, using inequalities (1.6) and (1.7) we get

$$\|¢ - «\| \leq 2g\left(\frac{1}{2}\|z - (STFG)z\|, \|z - (STFG)z\|, \|w - (STFG)w\|\right)$$

Since $\|¢ - «\| = 2\|w - (STFG)w\|$, so that above inequality gives

$$\|w - (STFG)w\| \leq g\left(\frac{1}{2}\|z - (STFG)z\|, \|z - (STFG)z\|, \|w - (STFG)w\|\right)$$

so that

$$\|w - (STFG)w\| \leq h/2\|z - (STFG)z\|$$

Thus from Lemma (1.4.1), we obtain

$$\|A^2 z - Az\| \leq h/2\|Az - z\|$$

Thus, Inductively we obtain

$$\|A^{n+1}z - A^n z\| \leq (h/2)^n \|Az - z\|$$

Since $h < 2$, it follows that $\|A^{n+1}z - A^n z\| \to 0$ as $n \to \infty$. Thus $\{A^n x\}$ is a Cauchy sequence and converges, to some point x_0 is X. We obtains, therefore $Ax_0 = x_0$ and so $(STFG)\, x_0 = x_0$.

So $(STFG)$ has at least one fixed point say x_0 in X i.e., $(STFG)\, x_0 = x_0$. Now using $(ST)^2 = I$, we get $FGx_0 = STx_0$ i.e. is a coincidence point of ST and FG. Now

$$\|STx_0 - x_0\| = \|STx_0 - ST(FGx_0)\|$$

$$\leq g\left(\|FGx_0 - FG(STx_0)\|, \|FGx_0 - STx_0\|, \|FG(STx_0) - ST(STx_0)\|\right)$$

$$< g\left(\|STx_0 - x_0\|, 0, 0\right)$$

$$< h\left(\|STx_0 - x_0\|\right)$$

yielding thereby $ST\,x_0 - x_0 = 0$, or $ST\,x_0 = x_0$ i.e., x_0 is a fixed point of ST and hence of FG.

To prove the uniqueness of common fixed point x_0, let y_0 be another fixed point of *ST* and *FG*, then

$$\begin{aligned} \|x_0 - y_0\| &= \|STx_0 - STy_0\| \\ &\le g\left(\|FGx_0 - FGy_0\|, \|FGx_0 - STx_0\|, \|FGy_0 - STy_0\|\right) \\ &< g\left(\|x_0 - y_0\|, 0, 0\right) \\ &< h\left(\|x_0 - y_0\|\right) \end{aligned}$$

giving thereby $x_0 - y_0 = 0$ i.e. x_0 is a unique common fixed point of *ST* and *FG*.

Now using the commutativity of the pairs (*F*,*G*), (*S*,*T*), (*FG*,*S*), (*FG*,*T*), (*ST*,*F*), (*ST*,*F*) and (*ST*,*G*) at x_0 one can write.

$Sx_0 = S(TSx_0) = ST(Sx_0)$, $Fx_0 = F(GFx_0) = FG(Fx_0)$,
$Tx_0 = T(TSx_0) = ST^2x_0 = ST(Tx_0)$, $Gx_0 = G(GFx_0) = FG(Gx_0)$,
$Sx_0 = S(FGx_0) = FG(Sx_0)$, $Fx_0 = F(STx_0) = ST(Fx_0)$,
$Tx_0 = T(FGx_0) = FG(Tx_0)$, $Gx_0 = G(STx_0) = ST(Gx_0)$,

which show that Fx_0, Gx_0, Sx_0 and Tx_0 is a common fixed point of the pair (*ST*,*FG*) which due to uniqueness of the common fixed point of the pair (*ST*,*FG*) get us.

$x_0 = Sx_0 = Tx_0 = Fx_0 = Gx_0$

This completes the proof.
After putting *FG* =*I* and *S* = *I*, in Theorem 1.1.4, we get the following result.

Corallary :

Let *T* be self mappings of a Banach space *X* satisfying

(i) $T^2 = I$,
(ii) $\|Tx - Ty\| \le g\left(\|x-y\|, \|x-Tx\|, \|y-Ty\|\right)$

for every $x, y \in X$ where $g \in ₵$, then *T* has at least one fixed point

Remark :

The foregoing Theorem 1.1.4 can be conveniently used to corollarize the theorem of Iseki (see[*]) if we choose $g(a,b,c) = (\sim / 2 + \quad) \max\{2a, b, c\}$ for all $a, b, c \ge 0$.

Now, in our next theorem we generlized the contractive condition given by Imdad and Khan [13].

Theorem :

Let *F*,*G*,*S* and *T* be self mappings of *a* Banach space *X* satisfying

(i) The pair (*ST*,*FG*) commute,
(ii) The pairs (*S*,*T*) and (*F*,*G*) are composite involutions,

(iii) $$\|STx - STy\| \le \frac{h}{2}\max\left(\|FGx - FGy\|, \frac{1}{2}\|FGx - STx\|, \frac{1}{2}\|FGy - STy\|, \frac{1}{2}\|FGx - STy\|, \frac{1}{2}\|FGy - STx\|\right) \quad ....(1.8)$$

for every $x, y \in X$ where $0 \le h < 4,$ then FG and ST have a coincidence point x_0 i.e., $FGx_0 = STx_0$. Moreover if the pairs $(S,T),$ (ST,G) $(ST,F),$ $(F,G),(FG,S)$ and (FG,T) commute at the foregoing fixed point x_0, then x_0 also remains the unique common fixed point of $S,T,$ F and G.

Proof : From (i) and (ii) it follows that $(STFG)^2 = I$. Now using (1.8), we have

$$\|STFGFx - STFGFy\| \le \frac{h}{2}\max\left(\|(FG)^2 Fx - (FG)^2 Fy\|, \frac{1}{2}\|(FG)^2 Fx - (STFG)Fx\|, \frac{1}{2}\|(FG)^2 Fy - (STFG)Fy\|, \frac{1}{2}\|(FG)^2 Fx - (STFG)y\|, \frac{1}{2}\|(FG)^2 Fy - (STFG)Fy\|\right)$$

If we set $Fx = z$ and $Fy = w$, then we get

$$\|STFGz - STFGw\| \le \frac{h}{2}\max\left(\|z - w\|, \frac{1}{2}\|z - (STFG)z\|, \frac{1}{2}\|w - (STFG)w\|, \frac{1}{2}\|z - (STFG)w\|, \frac{1}{2}\|w - (STFG)z\|\right).$$

Since the map $STFG$ is an involution and $0 \le h < 4,$ therefore by Theorem 2.1 (due to Khand and Imdad [13]), $STFG$ has at least one fixed point say x_0 in X i.e., $STFG\, x_0 = x_0$. Now using $(ST)^2 = I$, we get $FGx_0 = STx_0$ i.e. x_0 is a coincidence point of ST and FG. Now

$$\|STx_0 - x_0\| = \|STx_0 - ST(FGx_0)\|$$

$$\le \frac{h}{2}\max\left(\|FGx_0 - FG(STx_0)\|, \frac{1}{2}\|FGx_0 - STx_0\|, \frac{1}{2}\|FG(STx_0) - ST(STx_0)\|, \frac{1}{2}\|FGx_0 - ST(STx_0)\|, \frac{1}{2}\|FG(STx_0) - STx_0\|\right)$$

$$\le \frac{h}{2}\|STx_0 - x_0\|$$

yielding thereby $STx_0 - x_0 = 0,$ *or* $STx_0 = x_0$ *i.e.* x_0 is a fixed point of ST and hence of FG.

To prove the uniqueness of common fixed point x_0. Let y_0 be another fixed point of *ST* and *FG*, Then

$$\|x_0 - y_0\| = \|STx_0 - STy_0\|$$

$$\leq \frac{h}{2}\max\Big(\|FGx_0 - FGy_0)\|, \frac{1}{2}\|FGx_0 - STx_0\|, \frac{1}{2}\|FGy_0 - STy_0)\|,$$

$$\frac{1}{2}\|FGx_0 - STy_0\|, \frac{1}{2}\|FGy_0 - STx_0\|\Big)$$

$$\leq \frac{h}{2}\|x_0 - y_0\|$$

giving thereby $x_0 - y_0 = 0$ *or* $x_0 = y_0$ *i.e,* x_0 is a unique common fixed point of *ST* and *FG*.

Now using the commutativity of the pairs (*F*,*G*), (*S*,*T*), (*FG*,*S*), (*FG*,*T*), (*ST*,*F*) and (*ST*,*G*) at x_0 one can write.

$Sx_0 = S(TSx_0) = ST(Sx_0)$, $Fx_0 = F(GFx_0) = FG(Fx_0)$,
$Tx_0 = T(TSx_0) = ST^2x_0 = ST(Tx_0)$, $Gx_0 = G(GFx_0) = FG(Gx_0)$,
$Sx_0 = S(FGx_0) = FG(Sx_0)$, $Fx_0 = F(STx_0) = ST(Fx_0)$,
$Tx_0 = T(FGx_0) = FG(Tx_0)$, $Gx_0 = G(STx_0) = ST(Gx_0)$,

which show that Fx_0, Gx_0, Sx_0 and Tx_0 is a common fixed point of the pair (*ST*,*FG*) which due to uniqueness of the common fixed point of the pair (*ST*,*FG*) get us.

$$x_0 = Sx_0 = Tx_0 = Fx_0 = Gx_0$$

This completes the proof.

If we take $FG = I$ and $S = I$ in Theorem 1.1.7, we get the following result of Khan and Imdad [13].

Corollary :

Let *T* be self mappings of a Banach space *X* satisfying

(i) $T^2 = I$

(ii) $\|Tx - Ty\| \leq \frac{h}{2}\max\Big(\|x - y\|, \frac{1}{2}\|x - Tx\|, \frac{1}{2}\|y - Ty\|, \frac{1}{2}\|x - Ty\|, \frac{1}{2}\|y - Tx\|\Big)$

for every $x, y \in X$ where $0 \leq h < 4$, then *T* has at least one fixed point.

Remark :

Theorem 1.1.7, remains true if we replace condition 1.8) as follows

$\|STx - STy\| \leq h\|FGx - FGy\|$ for every $x, y \in X$, where $0 \leq h < 2$

We furnish an example to demonstrate the validity of the Remark 1.1.9

Example :

Let R be the set of reals equipped with usual norm. Define $S,T,F,G:R \to R$ as

$$Sx = \begin{cases} -x & if \quad x \ge 0 \\ -x/3 & if \; x < 0 \end{cases}, \quad Tx = \begin{cases} 3x & if \quad x \ge 0 \\ x & if \quad x < 0 \end{cases}$$

$$Fx = \begin{cases} -x & if \quad x \ge 0 \\ -x/4 & if \; x < 0 \end{cases}, \quad Gx = \begin{cases} 4x & if \quad x \ge 0 \\ x & if \quad x < 0 \end{cases}$$

So that

$$STx = \begin{cases} -3x & if \quad x \ge 0 \\ -x/3 & if \; x < 0 \end{cases} \quad \text{and} \quad FGx = \begin{cases} -4x & if \quad x \ge 0 \\ -x/4 & if \; x < 0 \end{cases}$$

Note that $(ST)^2 = (FG)^2 = I$

Now we distinguish following cases:

(a) For $x \ge 0, y \ge 0$ we have

$$\|STx - STy\| = 3|x - y| \le \frac{7}{8}(8|x - y|) = \frac{14}{8}(4|x - y|) = \frac{14}{8}\|FGx - FGy\|$$

(b) For $x < 0, y < 0$ we can write

$$\|STx - STy\| = \frac{1}{3}|x - y| \le \frac{7}{16}|x - y| = \frac{14}{8}\|FGx - FGy\|$$

(c) Next, for $x \ge 0 \; and \; y < 0$ we write a sequence of implications in the following way:

$$y < 0 \le x \Rightarrow y < \left(\frac{192}{5}\right)x \Rightarrow y < \left(\frac{48}{5}\right)4x \Rightarrow \left(\frac{5}{48}\right)y < 4x$$

$$\Rightarrow \frac{7}{6}y - \frac{1}{3}y < 7x - 3x \Rightarrow 3x - \frac{1}{3}y < 7x - \frac{7}{16}y = \frac{14}{8}\left|4x - \frac{y}{4}\right|$$

Which implies that

$\|STx - STy\| = \left|3x - \frac{y}{3}\right| \le \frac{14}{8}\left|4x - \frac{y}{4}\right| = \frac{14}{8}\|FGx - FGy\|$ Thus all the conditions of Remark 1.1.9 are satisfied if we choose $h = \frac{14}{8}$. Here $x = 0$ is the only coincidence point of ST and FG.

However 0 also remains the unique common fixed point of F,G,S and T.

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